Optimal staging of endoreversible heat engines

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(Received 25 February 1981; accepted for publication 19 August 1981)

One way to classify performance indices of irreversible heat engines is according to how the indices change when one engine is replaced by two (or more) of the same kind in series. We investigate the performance of two endoreversible engines (i.e., heat engines with the only irreversibility being heat resistance to the surroundings) which are put in series to form a single engine, whose power output is maximized. In this unconstrained optimization the interface between the two stages, which for the present model is the intermediate temperature and the relative timing of the two engines, is arbitrary and can be used to satisfy other, nonthermodynamic constraints. Adding any constraint on the volume of the working gas does not lift this indeterminacy. The optimum composite system is equivalent to a single endoreversible engine, thus displaying a sequencing property similar to Carnot engines.

PACS numbers: 05.70.Ln, 05.70.Ce

I. INTRODUCTION

The bounds on performance criteria in classical thermodynamics are provided by reversible processes. These bounds are of great interest because they are generally independent of the particular machinery employed to perform the processes which attain the bounds. The general study of irreversible engines has in part been stimulated by the fact that classical reversible thermodynamics provides bounds that are usually too optimistic to be useful benchmarks for real processes. The problem, of course, is that the inevitable losses associated with operating machinery at nonvanishing rates are neglected. Several previous studies[1–8] have attempted to repair this situation by incorporating the most important losses into the reference process.

In addition to the bounds there are other features of reversible processes which require reexamination when irreversibilities are considered. For example, the maximum efficiency for extracting work from a system composed of two heat reservoirs is the same whether one uses a single Carnot engine or several Carnot engines in series, parallel, or any combination thereof. In this paper we examine how performance criteria are affected for a class of irreversible engines when one such engine is replaced by two of these engines in series, by which we mean that the low-temperature reservoir of one engine is the high-temperature reservoir of the next (Fig. 1).

Unlike reversible engines, irreversible engines generally must be operated differently when different performance indices are optimized. Our purpose in studying the sequencing of irreversible engines was originally to see if it was possible to classify performance indices by their sequencing properties. We had expected different behavior depending on whether, for example, we maximized efficiency or average power output. As will be shown in this paper, this did not happen. Despite the more complex behavior of irreversible engines, they display a sequencing property similar to Carnot engines.

The class of engines we shall study are endoreversible[9] heat engines where the only irreversibility is due to heat resistances between the heat reservoirs and the reversible (e.g., Carnot) engine. We call such engines CA engines because the efficiency at maximum-average-power output was first calculated by Curzon and Ahlborn.[1] Complete, unconstrained optimizations of engines with only heat resistance losses, but without assuming a specific work cycle, were carried out in Refs. 5 and 6 for systems without and with volume constraints. In the present paper we investigate the optimal staging of two CA engines. There is some practical inspiration for this model from machines like two-stage refrigerators, which are usually employed to make liquid air,[10] multistage compressors for high pressure, and power plant turbines, which are often sequenced for high, medium, and low pressure.

We should emphasize that is not an unrestricted optimization (as in Refs. 5 and 6) since each of the engines are required to be of CA type. We use the CA-engine because it is the simplest irreversible engine. Furthermore, the CA-engine is not as unrealistic as it may appear at first sight. For example, the requirement that the isentropic branches occur in negligible time means that they must occur on a time scale that is fast compared to the slow rates for heat leaks to the environment but slow compared to the rapid internal relaxation of pressure gradients in the working fluid. An analysis of reversible, adiabatic branches may be found in Ref. 11.

The plan of this paper is as follows. In Sec. II we review the results for a single CA engine and consider a simple example of the sequencing of two CA engines. This example is made simple because the reservoir connecting the two engines is taken to have a constant temperature. In Sec. III, we present the detailed specification of the model we study. In Secs. IV and V the optimal cycle for maximum average power is computed. For those not interested in the mathematical details the results are presented and discussed starting with Eq. (62). In Sec. VI we consider the performance criteria of

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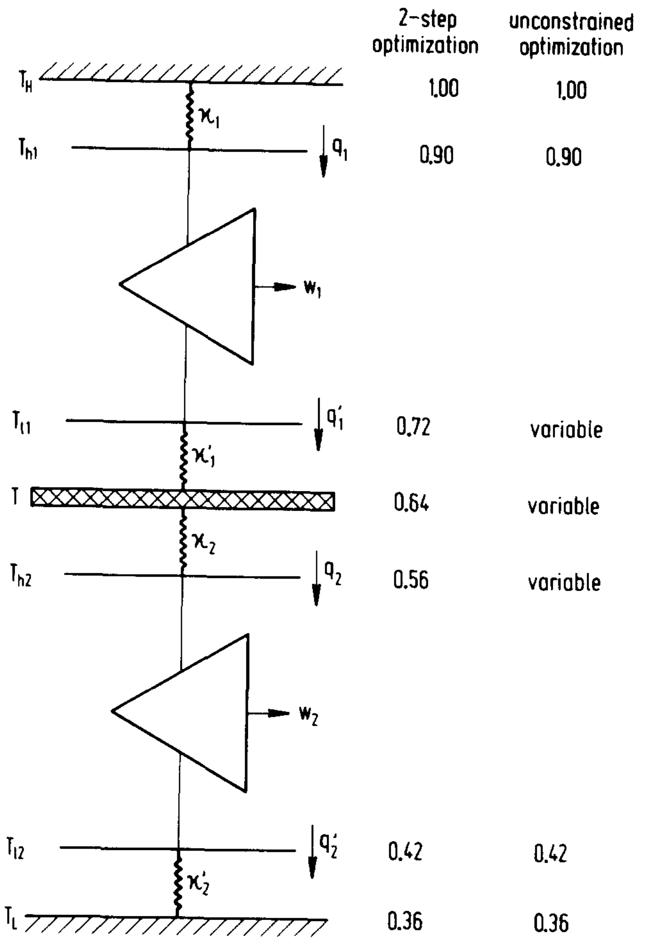

FIG. 1. Two staged CA engines. In the numerical examples T_H and T_L are the same as in Fig. 2. The first column results from the staging of two individually optimized CA engines with the intermediate temperature T a constant. The second column results from an unconstrained optimization of the whole system.

maxium efficiency, maximum effectiveness, and minimum entropy production. Finally Sec. VII contains our conclusions.

II. A SIMPLE EXAMPLE OF SEQUENCING

For comparison let us recapitulate the results for a single CA engine operating at maximum power[1] between reservoirs at temperatures T_H and T_L (see Fig. 2). The actual operating temperatures for the reversible cycle are

$$T_h = \tfrac{1}{2}T_H(1+x), \qquad T_l = \tfrac{1}{2}T_L(1+x^{-1}), \tag{1}$$

where

$$x = (T_L/T_H)^{1/2}, \tag{2}$$

when the heat conductances κ and κ' are the same. The corresponding thermal efficiency is calculated to be

$$\eta_0 = 1 - x = 1 - (T_L/T_H)^{1/2} \tag{3}$$

at the maximum-average-power output

$$w_{\max} = \tfrac{1}{4}\kappa T_H(1-x)^2 = \tfrac{1}{4}\kappa(\sqrt{T_H} - \sqrt{T_L})^2, \tag{4}$$

where the engine is in contact with each reservoir for half the cycling period, and the adiabatic branches have a negligible duration. Note the resemblance between Eq. (3) and the Carnot efficiency $\eta_{\max} = 1 - T_L/T_H$. A numerical example with $T_H = 1.00$ and $T_L = 0.36$ is indicated on Fig. 2.

Next let us put two CA engines on top of each other, separated by a reservoir at temperature T (see Fig. 1), but requiring that, on the average, this reservoir does not act as a heat source or sink. For simplicity we again take all the heat conductances equal and assume each engine operates at maximum average power output. Then, if the average rate of heat withdrawn from T_H is q_1, and we define

$$y = (T/T_H)^{1/2}, \tag{5}$$

the average power output of the top engine is

$$w_1 = (1-y)q_1. \tag{6}$$

Since the average heat flow out of the top engine and into the bottom engine is yq_1, the average power output of the second engine is

$$w_2 = (1 - x/y)(yq_1) = (y - x)q_1, \tag{7}$$

since $x/y = (T_L/T)^{1/2}$. Thus the the total average power production is

$$w = w_1 + w_2 = (1-x)q_1. \tag{8}$$

Note that Eq. (8) leads to the same efficiency as a single CA engine operated between the reservoirs T_H and T_L. This is just the same results one finds for Carnot engines; there is no change in efficiency if a single engine operating between two given reservoirs is replaced by staged engines. In the Appendix we show that this result follows simply from the fact that the output of one engine is the input of the second engine and the fact that at maximum power output $1-\eta$ is a ratio of the form $F(T_L)/F(T_H)$.

The intermediate temperatures can be calculated from analogs of Eq. (1) and the requirement that

$$q_1' = \frac{\kappa}{4}T(y^{-1} - 1) \tag{9}$$

equals

$$q_2 = \frac{\kappa}{4}T(1 - x/y). \tag{10}$$

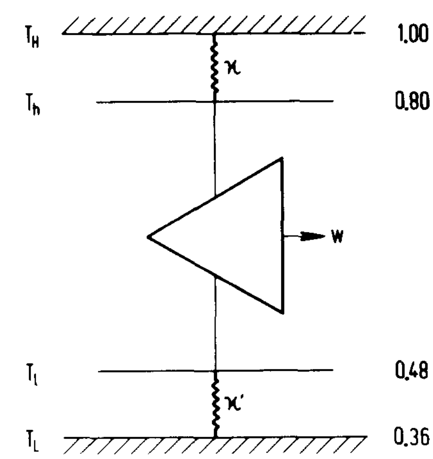

FIG. 2. A single CA engine operated between reservoirs T_H and T_L. In the example at right we have taken $T_H = 1.00$ and $T_L = 0.36$.

The results are

$$T_{h1} = \tfrac{1}{4}T_H(3+x), \quad T_{h2} = \tfrac{1}{8}T_H(1+x)(1+3x),$$
$$T_{l1} = \tfrac{1}{8}T_H(1+x)(3+x), \quad T_{l2} = T_H x(1+3x), \qquad (11)$$
$$T = \tfrac{1}{4}T_H(1+x)^2.$$

If we now calculate q_1 for the staged engines we find

$$q_1 = \frac{\kappa}{8}T_H(1-x), \qquad (12)$$

which is half the result for a single engine with the same cycling time. Thus the staged engines generate half the average power or, what is the same thing, do half the work in one period that the single engine does when they have the same heat conductances κ. This factor is easy to understand. If we call the thermal resistance of a CA engine $1/\kappa_h + 1/\kappa_l$, where κ_h (κ_l) is the heat conductance at the high-(low)temperature end of the engine, then if $\kappa_h = \kappa_l = \kappa$ for all engines, the staged engine has twice the thermal resistance that the single engine does.

The numerical values, with the same choice of reservoir temperatures as in Fig. 2, are indicated in the first column on Fig. 1. The reason for this little exercise, besides the interesting result of unchanged efficiency [Eq. (8)], is to provide a comparison for the less restricted results of the following sections, where all the intermediate temperatures are optimized simultaneously and permitted to vary in time.

Finally, note that with the exception of one temperature, T_H say, only temperature ratios enter into the calculation. As a consequence, an arbitrary number of engines may be squeezed in between T_H and T_L. The overall efficiency will remain unchanged, see Appendix, but the average power generated will decrease.

III. MODEL

The system we want to study is depicted in Fig. 1. It consists of two CA engines operating between the fixed-temperature reservoirs at T_H and T_L and coupled through the intermediate reservoir at $T(t)$. The important difference from the system described in Sec. II is that this reservoir has a finite heat capacity C, and its temperature may therefore vary in time as the CA engines go through their cycles. The heat conductances linking the reservoirs to the reversible engines are all constant. Each of the CA engines are assumed to contain a reversible (e.g., Carnot) engine with is in contact with the hot reservoir (through κ_i, i = 1 or 2) for the period t_i and in contact with the cold reservoir (through κ_i') for the period $\tau - t_i$, so that the total cycle time for both is τ. However, cycle 2 is shifted t_0 in time relative to cycle 1. The adiabatic branches, not slowed down by heat transfer, are assumed to transpire in zero time. Thus one period is divided into four segments:

(1) $t\epsilon[0,t_0[$: engine 1 connected to T_H, engine 2 to T_L,
(2) $t\epsilon[t_0,t_1[$: engine 1 connected to T_H, engine 2 to T,
(3) $t\epsilon[t_1,t_2+t_0[$: engine 1 connected to T, engine 2 to T,
(4) $t\epsilon[t_2+t_0,\tau[$: engine 1 connected to T, engine 2 to T_L.

(This is for $0 < t_0 < t_1$, as will be assumed in the rest of the paper. In case $t_1 < t_0 < \tau$ the segments come in the order 3, 1, 4, 2 without affecting any of our conclusions.) Figure 3 shows a possible time variation of the temperature of the intermediate reservoir, $T(t)$ and of the "isotherms" of the two reversible engines.

In the search for the time sequence which produces the maximum average power, $W/\tau = (W_1 + W_2)/\tau$, we keep the reservoir temperatures T_H and T_L and the cycle time τ fixed while letting all the other quantities vary: T_{h1}, T_{l1}, T, T_{h2}, T_{l2}, t_1, t_2, t_0. The variables are connected through the requirements of heat balance in the intermediate reservoir,

$$C\dot{T} = \kappa_1'(T_{l1} - T) - \kappa_2(T - T_{h2}), \qquad (13)$$

and reversibility of the internal engines,

$$\Delta S_1 = \int_0^{t_1} \kappa_1(T_H - T_{h1})/T_{h1}\,dt - \int_{t_1}^{\tau} \kappa_1'(T_{l1} - T)/T_{l1}\,dt = 0,$$
$$\Delta S_2 = \int_{t_0}^{t_2+t_0} \kappa_2(T - T_{h2})/T_{h2}\,dt - \int_{t_2+t_0}^{\tau} - \int_0^{t_0} \kappa_2'(T_{l2} - T_L)/T_{l2}\,dt = 0. \qquad (14)$$

The average power produced is $\overline{w} = (W_1 + W_2)/\tau$, where

$$W_1 = \int_0^{t_1} \kappa_1(T_H - T_{h1})dt - \int_{t_1}^{\tau} \kappa_1'(T_{l1} - T)dt,$$
$$W_2 = \int_{t_0}^{t_2+t_0} \kappa_2(T - T_{h2})dt - \int_{t_2+t_0}^{\tau} - \int_0^{t_0} \kappa_2'(T_{l2} - T_L)dt. \qquad (15)$$

The optimization will be carried out by the method of optimal control[12] with $W = W_1 + W_2$ as the objective function

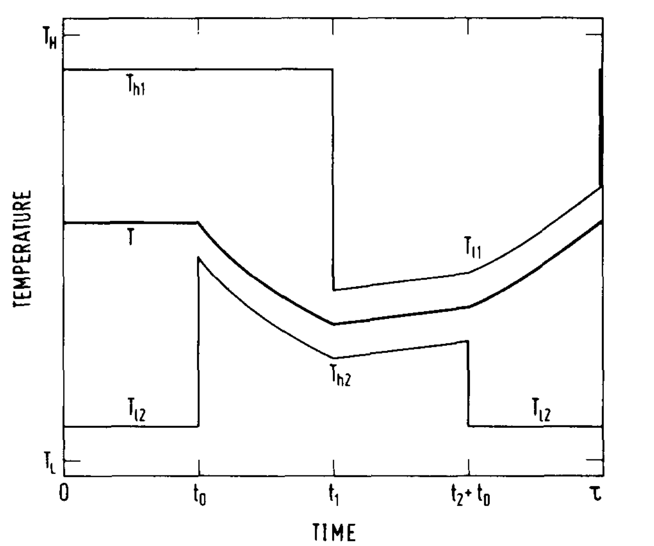

FIG. 3. Temperature sequence during one period for the system shown in Fig. 1 resulting from unconstrained optimization, i.e., corresponding to the last column of Fig. 1.

and Eqs. (13) and (14) as constraints, so that the Hamiltonian[12] of the problem H is defined by

$$\int_0^{\tau} H dt = W_1 + W_2 + \int_{t_0}^{\tau} \psi \kappa_1' (T_{l1} - T) dt - \int_{t_0}^{t_2 + t_0} \psi \kappa_2 (T - T_{h2}) dt - \lambda_1 \Delta S_1 - \lambda_2 \Delta S_2. \tag{16}$$

H is constant throughout the cycle. λ_1 and λ_2 are ordinary Lagrange multipliers corresponding to the integral constraints [Eq. (14)], whereas $\psi(t)$ is the adjoint variable associated with $T(t)$ via the differential constraint, Eq. (13). The connection between the two is the Hamiltonian equations

$$\dot\psi = -\frac{\partial H}{\partial T}, \qquad \dot T = \frac{\partial H}{\partial \psi}. \tag{17}$$

The remaining optimality equations are

$$\frac{\partial H}{\partial T_{h1}} = \frac{\partial H}{\partial T_{l1}} = \frac{\partial H}{\partial T_{h2}} = \frac{\partial H}{\partial T_{l2}} = 0, \tag{18}$$

which must be solved for each of the four time segments. When joining those into a complete cycle we can further take advantage of the continuity of $T(t)$ and $\psi(t)$ across the switching points.

We will for simplicity set $\kappa_1 = \kappa_1' = \kappa_2 = \kappa_2' = 1$ and $C = 1$ in the following derivation and only quote the general results at the end.

IV. SINGLE-BRANCH OPTIMIZATIONS

A. Branch 1, $t \in [0, t_0[$

Neither engine is connected to T during this time segment, so the Hamiltonian is simply

$$H = (T_H - T_{h1})(1 - \lambda_1/T_{h1}) - (T_{l2} - T_L)(1 - \lambda_2/T_{l2}) \tag{19}$$

and

$$\dot\psi = -\frac{\partial H}{\partial T} = 0, \tag{20}$$

$$\frac{\partial H}{\partial T_{h1}} = -1 + \lambda_1 (T_h/T_{h1}^2) = 0, \tag{21}$$

$$\frac{\partial H}{\partial T_{l2}} = -1 + \lambda_2 (T_L/T_{l2}^2) = 0,$$

which makes

$$\psi(t) = \text{const} \equiv \psi_0, \tag{22}$$

$$\lambda_1 = T_{h1}^2/T_H, \tag{23a}$$

$$\lambda_2 = T_{l2}^2/T_L \tag{23b}$$

Thus T_{h1} and T_{l2} are constant in this interval, and H becomes

$$H = (T_H - T_{h1})^2/T_H + (T_{l2} - T_L)^2/T_L = T_H(1 - R_1)^2 + T_L\left(\frac{1}{r_2} - 1\right)^2, \tag{24}$$

where we have introduced the dimensionless ratios

$$R_1 = T_{h1}/T_H, \qquad r_2 = T_L/T_{l2}. \tag{25a}$$

Later we will need the corresponding ratios

$$R_2 = T_{h2}/T, \qquad r_1 = T/T_{l1}. \tag{25b}$$

These ratios may be time dependent, but they are all less than one. Since the intermediate reservoir is isolated during this segment its temperature is constant, $T(t) = T_0$.

B. Branch 2, $t \in [t_0, t_1[$

Only engine 2 is connected to T, so

$$H = (T_H - T_{h1})(1 - \lambda_1/T_{h1}) + (T - T_{h2})(1 - \lambda_2/T_{h2}) - \psi(T - T_{h2}), \tag{26}$$

$$\dot\psi = -1 + \lambda_2/T_{h2} + \psi, \tag{27}$$

$$\frac{\partial H}{\partial T_{h1}} = 1 + \lambda_1 T_H/T_{h1}^2 = 0, \tag{28a}$$

$$\frac{\partial H}{\partial T_{h2}} = -1 + \lambda_2 T/T_{h2}^2 + \psi = 0. \tag{28b}$$

Eq. (28a) gives the same result for this branch as Eq. (23a),

$$\lambda_1 = T_{h1}^2/T_H = T_H R_1^2, \tag{29}$$

whereas Eq. (28b) can be solved for ψ,

$$\psi = 1 - \lambda_2 T/T_{h2}, \tag{30}$$

which simplifies the Hamiltonian to

$$H = T_H(1 - R_1)^2 + T_L \frac{1}{r_2^2}\left(\frac{1}{R_2} - 1\right)^2. \tag{31}$$

Since this must have the same (constant) value as in the previous branch, Eq. (24), the ratio R_2 must also be constant and equal to

$$R_2 = (2 - r_2)^{-1}. \tag{32}$$

Now Eq. (13) can be solved for $T(t)$,

$$\dot T = -T + T_{h2} = -\nu T, \tag{33}$$

where

$$\nu = 1 - R_2, \tag{34}$$

to give

$$T(t) = T_0 e^{-\nu(t - t_0)} \tag{35}$$

and, through Eq. (30),

$$\psi(t) = 1 - (T_L/T_0)(1/r_2^2 R_2^2) e^{\nu(t - t_0)}. \tag{36}$$

C. Branch 3, $t \in [t_1, t_1 + t_0[$

Here both engines are connected to T, and arguments analogous to the ones we have just used yield

$$r_1 = \text{const}, \qquad R_2 = \text{const}, \tag{37}$$

$$T_{h2}^2/T_{l1}^2 = r_1^2 R_2^2 = \lambda_2/\lambda_1, \tag{38}$$

$$T(t) = T(t_1) e^{-\gamma(t - t_1)}, \tag{39}$$

$$\psi(t) = 1 - [T_H/T(t_1)]r_1^2R_1^2e^{\gamma(t-t_1)}$$
$$= [T_L/T(t_1)]r_2^{-2}R_2^{-2}e^{\gamma(t-t_1)}, \quad (40)$$

where

$$\gamma = 2 - (1/r_1) - R_2. \quad (41)$$

Using our earlier definition of x in Eq. (2), Eq. (38) assumes the very symmetric form

$$x = r_1R_1r_2R_2. \quad (42)$$

D. Branch 4, $t\epsilon[t_2 + t_0,\tau[$

This branch is very similar to branch 2 with the roles of the engines interchanged. The solutions are

$$\lambda_2 = T_{l2}^2/T_L = T_L/r_2^2, \quad (43)$$

$$r_1 = \text{const} = 2 - (1/R_1), \quad (44)$$

$$T(t) = T(t_2 + t_0)e^{-\mu(t-t_2-t_0)}, \quad (45)$$

$$\psi(t) = 1 - \frac{T_H}{T(t_2+t_0)}r_1^2R_1^2e^{\mu(t-t_2-t_0)}, \quad (46)$$

with

$$\mu = 1 - (1/r_1). \quad (47)$$

V. CLOSING OF CYCLE

The solutions on the four time segments may now be joined into a complete cycle by invoking the continuity of $T(t)$ and $\psi(t)$ across the switching points. From this requirement we deduce that

(i) the integration constant T_0 of Eq. (35) is the same as the constant temperature T_0 in segment 1.

(ii) ψ_0, Eq. (22) can be found from Eq. (36),

$$\psi_0 = \psi(t_0) = 1 - (T_L/T_0)r_2^{-2}R_2^{-2}. \quad (48)$$

(iii) $T_{h2}(t)$ is continuous at t_1. This is not a trivial statement, since only the state functions $T(t)$ and $\psi(t)$ are required to be continuous, whereas the other functions in principle could contain jumps, caused by jumps in the control functions.

(iv) $T_{l1}(t)$ is continuous at $t_2 + t_0$.

The constancy of H, λ_1 and λ_2 have already been used several times. It should be emphasized that the constancy of H ensures the continuity of T_{h1} at t_0 and of T_{l2} at τ and thus makes all the ratios r_1, R_1, r_2, R_2 constant in their respective intervals of applicability. Finally, the periodicity and continuity of $T(t)$ implies that

$$T(0+) = T(\tau-),$$

or

$$T_0 = T_0e^{-\nu(t_1-t_0)}e^{-\gamma(t_2+t_0-t_1)}e^{-\mu(\tau-t_2-t_0)} \quad (49)$$

and gives a relation between t_1 and t_2

$$\nu(t_1 - t_0) + \gamma(t_2 + t_0 - t_1) + \mu(\tau - t_2 - t_0)$$
$$= (1 - R_2)(t_1 - t_0) + [2 - (1/r_1) - R_2](t_2 + t_0 - t_1)$$
$$+ [1 - (1/r_1)](\tau - t_2 - t_0)$$
$$= (1 - R_2)t_2 + [1 - (1/r_1)](\tau - t_1) = 0. \quad (50)$$

Note that both T_0 and t_0 have dropped out of this equation.

Our four temperature ratios are connected through Eqs. (32), (42), and (44), so all the expressions can be given in terms of one of them, say R_1. Then

$$r_1 = 2 - 1/R_1, \quad (44)$$

$$r_2 = 2x/(x - 1 + 2R_1), \quad (51)$$

$$R_2 = \tfrac{1}{2}[1 + x/(2R_1 - 1)], \quad (52)$$

and

$$\nu = \tfrac{1}{2}[1 - x/(2R_1 - 1)], \quad (53)$$

$$\gamma = \tfrac{1}{2}[3 - (2R_1 + x)/(2R_1 - 1)], \quad (54)$$

$$\mu = 1 - R_1/(2R_1 - 1), \quad (55)$$

while the adjoint variable is related to the temperature T in all time segments as follows,

$$\psi(t) = 1 - (2R_1 - 1)^2T_H/T(t). \quad (56)$$

A possible time variation for all temperatures involved is shown in Fig. 3, and a numerical example with T_H and T_L as specified in the Introduction is listed in the second column of Fig. 1.

At this point there are two equations left, Eqs. (14), which we use to determine t_1 and t_2.

$$\Delta S_1 = t_1(\frac{1}{R_1} - 1) - (\tau - t_1)(1 - r_1) = 0 \quad (57)$$

$$\Rightarrow t_1 = \tfrac{1}{2}\tau \quad (58)$$

and

$$\Delta S_2 = t_2(\frac{1}{R_2} - 1) - (\tau - t_2)(1 - r_2) = 0 \quad (59)$$

$$\Rightarrow t_2 = \tfrac{1}{2}\tau. \quad (60)$$

Combining this with Eq. (50) we finally obtain the last piece in our puzzle,

$$R_1 = \tfrac{1}{4}(3 + x), \quad (61)$$

from which all the other quantities may be calculated.

The average power produced by this optimal composite engine is $w = W/\tau$, where

$$W = \frac{\tau}{2}T_H(1 - R_1) - (\frac{1}{r_1} - 1)\int_{\tau/2}^{\tau}T(t)dt + (1 - R_2)$$
$$\times\int_{t_0}^{\tau/2+t_0}T(t)dt - \frac{\tau}{2}T_L(\frac{1}{r_2} - 1) = \frac{\tau}{8}T_H(1 - x)^2, \quad (62)$$

at efficiency

$$\eta = \frac{w}{q_1} = [\tfrac{1}{8}T_H(1 - x)^2]/[\tfrac{1}{8}T_H(1 - x)] = 1 - x, \quad (63)$$

which is exactly the CA efficiency (cf. the Introduction). This is a most remarkable result, especially when one notes that the final results are independent of T_0 and t_0. This means that the relative phase of the two engines (t_0) as well as the intermediate temperature, T (as long as it is between T_{h1} and T_{l2}, which are fixed) are immaterial. Even if one of the engines deviates from maximum power output, the other will exactly compensate for this (through a changed temperature range) to make the composite engine appear as a single CA engine. The staging of two CA engines, as described in Sec. II, is but one choice of the intermediate temperature T. It is worth repeating that the results presented above are for equal heat conductances, $\kappa_1, \kappa_1', \kappa_2, \kappa_2'$, which greatly simplifies the expressions and, e.g., makes $\gamma = 0$, i.e., all temperatures are constant in time segment 3 where both engines are connected to the intermediate reservoir.

The practical implications of this result are that

(1) There is no thermodynamic basis for choosing between a single-stage and a multi-stage machine with the same overall heat resistances, i.e., referring to Figs. 1 and 2,

$$\kappa^{-1} + \kappa'^{-1} = \kappa_1^{-1} + \kappa_1'^{-1} + \kappa_2^{-1} + \kappa_2'^{-1}, \tag{64}$$

for a fixed cycle time τ. Of course if one uses the maximum conductivity in all cases so all κ's are equal, then the right-hand side of Eq. (64) is twice as large as the left-hand side and, consequently, w is half as large for the sequenced engines as for the single engine.

(2) There is no thermodynamic basis for choosing the phasing or the intermediate temperature in a two-stage system (within the limits mentioned above).

Both questions must be settled by engineering preferences or limits such as size of the machinery, materials strength, and complexity.

One possible additional constraint which could be added to this problem to make it more realistic concerns the volume swept by the two engines, e.g., the total volume or the sum of the compression ratios. So far the "internal" engines (triangles in Figs. 1 and 2) have only been assumed to be reversible and exchange heat isothermally, but if we want to consider the volume behavior, we must specify the precise type of cycle used and the equation of state of the working fluid. Only the Carnot, Stirling, and Ericsson cycles qualify as exchanging heat isothermally (cf. the pressure-volume (PV) and temperature-entropy (TS) diagrams in Ref. 7). Let us for simplicity choose an ideal gas in a Carnot engine. Then

$$\begin{aligned} Q_1 &= \kappa_1 t_1 (T_H - T_{h1}) \\ &= n_1 R T_{h1} [\ln(V_{l1}/V_{h1}) - (\gamma - 1)^{-1} \ln(T_{h1}/T_{l1})], \end{aligned} \tag{65}$$

where n_1 is the number of moles of gas in the engine, V_{l1}/V_{h1} is the compression ratio, and $\gamma = C_p/C_v$ the ratio of heat capacities. A similar equation holds for engine 2, and it makes no difference that the branch in contact with T is not truly isothermal, because the other isotherm is, and two branches are sufficient to completely define the endpoints of the cycle. Eq. (65) can easily be solved for n_1 to correspond to any desired temperatures and compression ratio, so any volume constraint carries with it the freedom of adjusting the amount of working fluid, and T_0 and t_0 are still indeterminate. The choice must again be made on engineering grounds.

VI. OTHER OBJECTIVE FUNCTIONS

The above analysis was carried out in order to maximize the power output of the engine (as was done in Ref. 1). Other objective functions, like efficiency, effectiveness, and entropy production, can be treated in a completely analogous way by adding extra constraints to the Hamiltonian, Eq. (16). For example, maximizing the *efficiency* $\eta = W/Q_1$ can be done by maximizing W subject to fixed $Q_1 = t_1\kappa_1(T_H - T_{h1}) = \text{const}$. Doing that, the efficiency becomes

$$\eta = 1 - x^2 \Big/ \left(1 - \frac{8Q_1}{T_H \tau}\right). \tag{66}$$

Its natural limits are

$$\eta \to 1 - x^2 = \eta_{\text{rev}} \quad \text{for } Q_1 \to 0 \quad \text{or } \tau \to \infty, \tag{67}$$

and

$$\eta \to -\infty \quad \text{for } Q_1 \to T_H \tau/8, \tag{68}$$

so that the efficiency approaches the Carnot efficiency for very slow operation and decreases for increased heat input. For a single engine a similar optimization yields

$$\eta = 1 - x^2 \Big/ \left(1 - \frac{4Q_1}{T_H \tau}\right). \tag{69}$$

Thus the efficiency of the staged system is less than that for a single-stage engine operating between the same high- and low-temperature reservoirs with the same period *and* heat influx Q_1. The difference is the factor of 8, rather than 4 for the single stage, multiplying Q_1 in Eq. (66). For our example with $T_L/T_H = 0.36$ and taking $4Q_1/T_H\tau = 0.1$, the maximum efficiency of the single engine is $\eta_{\max} = 0.60$ while for the two-stage engine it is $\eta_{\max} = 0.55$. If in this example one takes $T/T_H = 0.64$, the efficiency of the two stages is $\eta_1 = 0.29$ and $\eta_2 = 0.37$, and $1 - \eta_{\max} = (1 - \eta_1)(1 - \eta_2)$ as required by Eq. (A4). Thus staging the engines while keeping the heat influx the same decreases the efficiency as would be expected from the increase in thermal resistance.
be expected from the increase in thermal resistance.

The *effectiveness*, $\epsilon = W/W_{\text{rev}}$ difffers, in this case, from the efficiency only by a constant, since

$$W_{\text{rev}} = Q_1(1 - x^2), \tag{70}$$

and its optimal behavior is identical to the one just quoted for η.

It is easy to show that minimizing the *entropy production* per cycle for fixed Q_1 is the same as maximizing the effectiveness for fixed Q_1. To see this we observe that the total entropy production per cycle is

$$\Delta S = \frac{Q_2}{T_L} - \frac{Q_1}{T_H}. \tag{71}$$

Since the actual work done per cycle is $W = Q_1 - Q_2$,

$$\Delta S = -\frac{W}{T_L} + \frac{Q_1}{T_L}\left(1 - \frac{T_L}{T_H}\right). \tag{72}$$

Using Eq. (70) and the definition of effectiveness, we then find

$$\begin{aligned} \Delta S &= (W_{\text{rev}} - W)/T_L \\ &= (1 - \epsilon) W_{\text{rev}}/T_L, \end{aligned} \tag{73}$$

where W_{rev} and T_L are constant, so that minimum ΔS corresponds to maximum ϵ.

VII. CONCLUSIONS

Using a model for a two-stage thermal engine, we have determined the configuration which produces the most power. We have found that the interface between the two stages (temperature and timing) is, within limits, arbitrary and can be used to satisfy other, nonthermodynamic constraints. The building block in finite-time thermodynamics is the CA engine, which in many respects behaves similarly to the Carnot engine of reversible thermodynamics. Thus CA engines can be staged to form an overall CA engine, and the unconstrained staging optimization also yields CA behavior. This is the first attempt to look at composite systems in finite-time thermodynamics and should facilitate its application to more complex systems based on the models solved so far.

ACKNOWLEDGMENTS

We would like to thank the Aspen Center for Physics for providing the excellent surroundings which brought us together and stimulated us during this work. B.A. also wants to thank NATO for a travel grant.

APPENDIX

In this appendix, we provide a derivation of the result of Sec. II in a simple way that illustrates the importance of the heat output and input of successive engines being equal and of the form of $1-\eta$.

Consider a sequence of N engines. The efficiency of the j^{th} engine is

$$\eta_j = 1 - Q_j'/Q_j, \quad \text{(A1)}$$

where Q_j is the heat input per cycle, and Q_j' is the heat ouput per cycle. Now suppose

$$Q_j' = Q_{j+1} \quad \text{for } j = 1, 2, \ldots, N-1. \quad \text{(A2)}$$

Then

$$(1-\eta_1)\ldots(1-\eta_N) = Q_N'/Q_1. \quad \text{(A3)}$$

But the efficiency of the entire engine is just given by $\eta = 1 - Q_N'/Q_1$, so

$$1-\eta = (1-\eta_1)\ldots(1-n_N). \quad \text{(A4)}$$

Next suppose that, if we optimize the operation of a single engine for some choice of objective function such as maximum average power output, that

$$1-\eta = f(T_L)/f(T_H), \quad \text{(A5)}$$

where T_H is the temperature of the high-temperature reservoir and T_L is the temperature of the low-temperature reservoir. Suppose we select a series of reservoir temperatures

$$T_H = T_1 > T_2 > \ldots > T_{N+1} = T_L, \quad \text{(A6)}$$

and operate each sequenced engine in its optimal mode so that

$$1-\eta_j = f(T_{j+1})/f(T_j), \quad \text{(A7)}$$

then Eq. (A4) implies that

$$1-\eta = f(T_{N+1})/f(T_1) = f(T_L)/f(T_H), \quad \text{(A8)}$$

That is, the efficiency of the sequenced engines equals that of a single engine. This is the case for Carnot engines where $f(T) = AT$ and CA engines, where $f(T) = B\sqrt{T}$. It does not follow that the work done per cycle is the same for the sequenced engines as for the single engine.

If η is not itself an extremum, then the individual engines in the sequenced case need not be operated optimally in order for the overall efficiency to be equal to $1-f(T_L)/f(T_H)$.) This occurs if the average power is maximized.[5] Thus there is some freedom in selecting the operation of the individual engines. If the objective function is the efficiency itself, then it is clear from Eq. (A4) that each engine must be operated at maximum efficiency; however, even in this case the intermediate reservoir temperatures may be selected arbitrarily provided they remain ordered according to Eq. (A6).

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