Multiple Modes for the Operation of a Binary Distillation Column

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We investigate whether it is possible to operate a given binary distillation column with several different reflux ratios (heat flow rates) while achieving the same purity and amount of distillate. We find that even for small variations in the plate efficiency vs flow rate several distinct values of the reflux ratio exist that achieve the same output of the column.

I. Introduction

Distillation has been a technological process of great interest for several millennia, and much heuristic, experimental, and theoretical knowledge about the behavior of distillation columns has been obtained [1–4]. Since distillation is such a common process in science and engineering, the question of efficiency and subsequent optimization of the distillation process has often been raised. Recent investigations are concerned with minimizing the entropy production in distillation [5–7] and with the control of distillation columns [8].

The field is still very much active, and the final answer is by no means found. Especially economic considerations have driven the quest for higher efficiency and optimal use of resources [1], and an economic optimization is always performed before a column is actually built and taken into operation.

However, for a given ordinary distillation column there remains for all practical purposes only one control parameter that can be adjusted, once the amount and purity of the feed and distillate have been prescribed. This quantity is the reflux ratio at the condenser. It translates into the amount of heat flow through the column which is proportional to the energy consumption and entropy production during the process.

Thus it is of great interest, not only for economic reasons but also for reasons of control, to know whether there might be several values of the heat flow (reflux ratio) that allow the operation of a given column while producing a desired purity of the distillate. In this paper we address this issue generally and show that fairly rapid changes in the plate efficiency vs flow rate, even of small magnitude, may result in the existence of multiple operating modes. A related study [9] of multiplicity in a multicomponent distillation was based on global material balances with several solutions.

After presenting qualitative arguments for the existence of such multiple operating modes in section II, we show their presence explicitly for the case of an ideal binary distillation column in sections III and IV. Finally, in section V, the effect of nonidealities is discussed, and comparison is made to other efficiency studies primarily based on entropy production.

II. Qualitative Arguments for the Existence of Multiple Operating Modes

A distillation process using perfect plates (or, at least, plates with constant efficiency) will, for a given number of plates N, in the column, exhibit only one possible reflux ratio R (heat flow Q) that can produce the desired distillate, since the number of required plates N(R) decreases monotonically with R [3]. The distillate flow D, the final purity x_f, and the initial feed concentration x_i are assumed to be fixed for a given process.

As indicated in the Introduction, the cause of the possible existence of multiple operation modes lies in the dependence of the plate efficiency on the heat and/or material flows. Clearly, if the plate efficiency approaches zero with increasing reflux ratio, then the number of plates necessary to achieve a certain amount of purified distillate will approach infinity in the limit of zero plate efficiency. Similarly, an infinite number of plates is necessary for reflux ratios approaching the minimum reflux ratio [3]. There must therefore exist a minimum in the number of required plates, N_{min}, for some reflux ratio between R_{min} and infinity. If the column is built with a number of plates N > N_{min}, then there are obviously two reflux ratios for which this column will produce the same purity.

However, the second solution will in general occur at such large values of the reflux ratio that one would never realistically contemplate running a column in that regime. Thus for practical purposes the question is “Can multiple solutions associated with minima in the curve N(R) (or N(Q)) occur in the regime of reflux ratios that is commonly employed for the economically optimal operation of the column?”

The answer to this question is affirmative. Any rapid change in the plate efficiency, even of small magnitude, can introduce a local minimum in the required number of plates vs reflux ratio (heat flow) curve for given fixed input and output concentrations. Since most distillation columns are constructed of discrete plates (stages), the number of necessary plates for the actual operation will necessarily change in steps of unity as the reflux ratio R is varied. Thus the actual curve N_{obs}(R) of necessary plates will be an integer step function approximating the continuous curve N(R). As a consequence many small local minima in N(R) may not be noticeable in N_{obs}(R) if they fall well within a single step of N_{obs}(R).

However, if the local minimum of the continuous N(R) curve happens to dip below an integer value, even ever so slightly, the integer N_{obs}(R) curve will also display two values of R corresponding to a particular number of plates, separated by a (small) region with a value of N_{obs}(R) one higher (cf. e.g. Figure 2 where the horizontal dashed line would correspond to that particular number of plates). These two reflux ratios (or the corresponding heat flows Q_1 and Q_2) constitute the multiple solutions mentioned above.

In order to go beyond these purely qualitative arguments, we will in the next two sections calculate N(Q) for an ideal distillation process with plates whose efficiency P(Q) decreases with increasing heat flow Q.
III. Equations Describing a Distillation Column

We are not interested in taking into account specific details of the materials that have to be separated nor general imperfections of distillation columns like pressure drops, heat leaks, etc. beyond including these design effects into the plate efficiency. For our purposes any perfect ideal distillation column can be described by demanding that mass conservation of the components and enthalpy conservation hold at each plate. We will consider the case of binary distillation of two ideally mixed substances. We assume no heat of mixing or radiation losses and a non-negligible pressure drop. In addition, we assume that all the plates exhibit the same plate efficiency as a function of heat flow, i.e. the small variations in liquid/vapor flow over the length of the column do not matter as far as the plate efficiency is concerned (see section V for the case that the plate efficiency depends on the material flows instead).

We count plates from the top, i.e. for the rectifying section the condenser is plate 0 and feed is on plate $n_F$, while for the stripping section 0 is the feed plate and $n_W$ corresponds to the boiler. Expressions with a caret refer to the stripping/rectification part of the column.

Perfect (plate efficiency equals one), ideal (eq 4c below applies) binary distillation column operation can be described by the following transfer equations (see Figure 1 for the traditional definition of the variables):

$$V_n + L_n = V_{n+1} + L_{n-1} \quad \text{(overall mass conservation)} \quad (1)$$

$$y_n V_n + x_n L_n = y_{n+1} V_{n+1} + x_{n-1} L_{n-1} \quad \text{(component one mass conservation)} \quad (2)$$

$$Q_n^\text{vapor} + Q_n^\text{liquid} = Q_{n+1}^\text{vapor} + Q_{n-1}^\text{liquid} \quad \Rightarrow \quad V_n y_n (H_1 + C_{V1} T_n) + (1 - y_n) (H_2 + C_{V2} T_n) + L_n (x_n C_{L1} T_n + (1 - x_n) C_{L2} T_n) = V_{n+1} y_{n+1} (H_1 + C_{V1} T_{n+1}) + (1 - y_{n+1}) (H_2 + C_{V2} T_{n+1}) + L_{n-1} (x_{n-1} C_{L1} T_{n-1} + (1 - x_{n-1}) C_{L2} T_{n-1}) \quad \text{(enthalpy conservation)} \quad (3)$$

$$T_n = T_{eq}(y_n) \quad (4a)$$

$$T_n = T_{eq}(x_n) \quad (4b)$$

$$y_n^{eq}(x_n) = \frac{\alpha x_n}{1 + (\alpha - 1)x_n} \quad \text{(vapor curve)} \quad (4c)$$

Equations 4a–c describe the equilibrium curves of the phase separation diagram of the ideal mixture made up of substances one and two. $\alpha$ is the relative volatility constant, and we choose component one to be purified, i.e. $\alpha > 1$. Note that only two of the three equations (4a–c) are independent.
Finally, the boundary conditions at the condenser, feed-plate, and boiler tie the variables to the exterior reservoirs and fix the overall heat flows:

\[ x_0 = y_1 = x_F, \quad V_1 = L_0 + D \quad (4d) \]

\[ \dot{x}_0 = x_n = x_F, \quad \dot{y}_0 = y_n = y_F \equiv y_0^N(x_0), \]

\[ V_n + L_0 = L_{n-1} + F + \dot{V}_1 \quad (4e) \]

\[ x_{n-1} = x_w, \quad y_{n-1} = y_w \equiv y_0^N(x_w), \quad \dot{V}_{n-1} = \dot{L}_{n-1} - W \quad (4f) \]

\[ Q_{\text{out}} + Q_{\text{distillate}} + Q_{\text{liquid}} = Q_{\text{vapor}} \quad (4g) \]

\[ Q_{\text{feed}} + Q_{\text{liquid}} - Q_{\text{vapor}} = Q_{\text{liquid}} + Q_{\text{vapor}} \quad (4h) \]

\[ Q_{\text{in}} + Q_{\text{liquid}} = Q_{\text{waste}} + Q_{\text{vapor}} \quad (4i) \]

The concentrations of component one in the distillate and the waste are given by \( x_0 \) and \( x_w \), respectively, while \( x_n \) and \( y_n \) refer to the concentrations in the liquid and vapor at the feed plate, respectively. \( D, W, \) and \( F \) are the distillate, waste, and feed (typically boiling liquid) flows, respectively. Finally, \( Q \) denotes the enthalpy flows carried by the various parts of the system; \( Q_{\text{in}} \) is the heat added at the boiler, while \( Q_{\text{out}} \) is the heat extracted at the condenser.

We can now integrate eqs 1–3 and get the constants

\[ V_{n+1} = L_n + D, \quad \dot{V}_{n+1} = \dot{L}_n - W \quad (5) \]

\[ y_{n+1}V_{n+1} = x_nL_n + x_DD, \quad y_{n+1}\dot{V}_{n+1} = x_n\dot{L}_n - x_wW \quad (6) \]

\[ V_n[y_nH_1 + C_{LV}T_n + (1 - y_n)H_2 + C_{LV}T_n] = L_n[x_nC_{L1}T_n + (1 - x_n)C_{L2}T_n] = Q \quad (\text{constant}) \quad (7) \]

\[ \dot{V}_n[y_nH_1 + C_{LV}T_n + (1 - y_n)H_2 + C_{LV}T_n] = \dot{L}_n[x_nC_{L1}T_n + (1 - x_n)C_{L2}T_n] = \dot{Q} \quad (\text{constant}) \quad (8) \]

\[ Q \quad \text{and} \quad \dot{Q} \quad \text{are conserved quantities of the system, since} \]

\[ y_n - y_{n+1} = P_n[y_0^N(x_n) - y_{n+1}] = P_n[y_0^N(x_n(y_{n+1}, V_{n+1}(y_{n+1}))) - y_{n+1}] = P_nf(y_{n+1}) \quad (11) \]

where

\[ f(y_{n+1}) = (y_n - y_{n+1})_{\text{perfect}} = \frac{\alpha[Qy_{n+1} - x_FD(cy_{n+1} + H)]}{[Q - D(cy_{n+1} + H)] + (\alpha - 1)[Qy_{n+1} - x_FD(cy_{n+1} + H)]} - y_{n+1} \quad (12) \]

A similar expression is found for the stripping section. Making the reasonable assumption that all plates have the same plate efficiency as a function of heat flow, we get

\[ y_n - y_{n+1} = P(Q)\left[\frac{\alpha[Qy_{n+1} - x_FD(cy_{n+1} + H)]}{[Q - D(cy_{n+1} + H)] + (\alpha - 1)[Qy_{n+1} - x_FD(cy_{n+1} + H)]} - y_{n+1}\right] \quad (13) \]

In order to proceed beyond this result we now recast eq 13 in continuous form, approximating \( y_n - y_{n+1} \) by \(-dy/dn\) to find

\[ -\frac{dy}{dn} = P(Q)\left[\frac{\alpha[Qy - x_FD(cy + H)]}{[Q - D(cy + H)] + (\alpha - 1)[Qy - x_FD(cy + H)]} - y\right] = P(Q)f(Q,y) \quad (14) \]
To illustrate the qualitative validity of eq 14 in view of these approximations, we have studied the distillation profile y(n). We find that y(n) decreases monotonically with n and that for most values of heat flow y(n) exhibits an inflection point between the feed and the condenser (see Appendix for details). This is in agreement with the distillation profile of many typical distillation processes [3].

Encouraged by this fact we now proceed with the evaluation of N(q), the number of required plates between the feed and the condenser. It is of course sufficient for our purposes to continue to deal only with the rectifying part of the column. We can integrate eq 14 for fixed heat flow q and fixed system/process parameters α, H, ε, x₀, xₚ (or yₚ = yₚ(xₚ)), and D. The exact result is

\[ N(q) = -\int_{y_f}^{\infty} \frac{dy}{P(q)f(y,q)} = -\frac{1}{P(q)} \int_{y_f}^{\infty} \frac{dy}{f(y,q)} \]

\[ = \frac{1}{P(q)} \left[ N^*(q) \right] = \frac{2}{P(q)} \left[ \frac{q_α + a_2}{w} \times \ln \left( \frac{q_α + a_7 + a_8w}{q_α + a_7 - a_8w} - \ln \left( \frac{q_α + a_10}{q_α + a_{12}} \right) \right) \right] \]

where \( w = (q_2^2 + q_α + a_3)^{1/2} \). Here \( a_1 \) through \( a_{12} \) are constants:

\[ a_1 = α + 1, \quad a_2 = -\left[ H[1 + (α - 1)x₀] + αx₀ \right], \]
\[ a_3 = (α - 1)^2, \quad a_4 = 2(α - 1)[H[1 + (α - 1)x₀] - αx₀], \]
\[ a_5 = q_α^2 + q_α x₀ + x₀^2 \]
\[ a_6 = (α - 1)(x₀ + y_f - 2x₀y_f), \]
\[ a_7 = [H(x₀ + y_f) + 2e x₀ y_f][1 + (α - 1)x₀] - αx₀[2H + ε(x₀ + y_f)], \]
\[ a_8 = (x₀ - y_f), \]
\[ a_9 = x₀(1 - x₀), \]
\[ a_{10} = -(H + εx₀)x₀(1 - x₀), \]
\[ a_{11} = y_f(1 - y_f), \]
\[ a_{12} = -q_{min}y_f(1 - y_f) = \frac{-q_{min}y_f(1 - y_f)}{α - 1} \]

The plate efficiency \( P(q) \) thus only enters in an average way by increasing \( N^*(q) \), the minimum number of plates for perfect trays (P = 1), proportionally. As expected, \( N^*(q) \) decreases monotonically with \( q \), with the limiting values

\[ \lim_{q \to \infty} N^*(q) = \frac{α}{α - 1} \ln \left( \frac{1 - y_f}{1 - x₀} \right) + \frac{1}{α - 1} \ln \left( \frac{x₀}{y_f} \right) \]
\[ \lim_{q \to q_{min}} N^*(q) = \infty \]

where

\[ q_{min} = (H + εy_f)\frac{[αx₀(1 - y_f) - y_f(1 - x₀)]}{y_f(1 - y_f)(α - 1)} = \frac{(H + εy_f)q_α}{y_f - x₀} = (H + εy_f)R_{min}^n + 1 \]

is the minimum heat flow (\( q_{min} > q_0 = εx₀ + H \)). We introduced the expression \( R_{min}^n \) in order to make contact with the standard definition of minimum reflux ratio which applies for the case \( ε = 0 \). In general

the minimum reflux ratio \( R_{min}^n \) is found by taking the maximum along the entire column of all the "local minimum reflux ratios" up to plate n [1],

\[ R_{min}^n = \frac{x₀ - y_n}{y_n - x_n} \]

(19)

For \( ε = 0 \) this maximum occurs at \( n_r \). In this case the reflux ratio is constant throughout the rectifying part of the column, and

\[ q_{min} = (R_{min}^n + 1)H \]

(20)

In general \( q_{min} \) can be uniquely related to the reflux ratio at any specified plate, e.g. the condenser, via eqs 4–7. An expression similar to eq 15 is found for the required number of plates in the stripping section.

**IV. Multiple Modes of Operation for a Given Column**

As noted in section III, if \( P(q) \to 0 \) for \( q \to \infty \), then \( N(q) = N^*(q)/P(q) \) will exhibit a global minimum in the interval \( (q_{min}, \infty) \). Therefore, the existence of multiple solutions is guaranteed if one considers the whole range of heat flows.

However, even in the realistic operating region, where both heat flow and required number of plates are small, a rapid change of plate efficiency, even if of small magnitude, will produce a local minimum in \( N(q) \) along with multiple solutions of the equation \( N(q) = N_c \).

A possible cause for such a change could be the entrainment of liquid droplets in the vapor [1]. We expect that this phenomenon would become important only above some critical value of the vapor flow. At the onset of entrainment the plate efficiency should decrease rather rapidly and then continue to decline at a slower rate until additional effects (e.g. excessive frothing) strongly reduce the plate efficiency, ultimately close to zero. However, we are only concerned with modeling the behavior of a column which is operated in the region where entrainment first sets in and where the plate efficiency shows a limited drop while remaining nearly constant before and after the change. Such a limited decrease in plate efficiency can conveniently be modeled by assuming that \( 1/P(q) \) has the following form in that particular region of heat flow values:

\[ \frac{1}{P(q)} = f_1 + f_2 \tanh((q - q_{drop})/r) \]

(21)

with constants \( f_1, f_2, r \).

In order to show that eq 21 can result in multiple solutions, we consider a numerical example of the separation of a mixture of 60 wt % benzene and 40 wt % toluene into 97 wt % benzene at the condenser with a distillate flow of \( D = 100 \text{ mol/s} \). The appropriate constants are [1]

\[ α = 2.4, \quad y_f = y_{qα}(x_f) = 0.65, \quad x₀ = 0.974, \]
\[ H = 7.24 \text{ kcal/mol} = 30.26 \text{ kJ/mol}, \quad ε = 0 \]

resulting in a minimum heat flow of \( q_{min} = 18.20 \text{ kcal/mol} = 76.08 \text{ kJ/mol} \). We will assume that the plate efficiency changes from 1.0 to 0.91 over a roughly 5% interval around a heat flow value of \( q_{drop} = 19 \text{ kcal/mol} = 79.42 \text{ kJ/mol} \) as illustrated on the right hand scale of Figure 2. The corresponding number of required plates...
In the preceding section we have calculated the number of plates required for a specific distillation process as a function of heat flow, \( N(q) \). It was shown explicitly that several values of heat flow \( q \) can exist which produce the same output of a given distillation column with a fixed number of plates \( N_c \). Of course, we must always address the question whether the model we have used in the calculation is sufficiently refined, i.e., could it happen that additional imperfections of the column and/or nonidealities of the mixture might invalidate our conclusions?

Let us consider the most important sources of potential problems:

(a) The plate efficiency decreases only very slowly for heat flows in the typical region of operation. Clearly, if in the region of interest \( P(q) \) decreases so slowly with \( q \) that the relative decrease of the number of plates is always outpacing the relative drop in the efficiency, \((dn*/dq)/n* < (dp/dq)/p \), then \( N(q) = N*/p \) decreases monotonically, and no local minimum exists. If this is the case, multiple modes are not expected.

(b) The mixture is nonideal, the heat of mixing is nonnegligible, or the specific heats contribute significantly to the enthalpy flow. If one of these problems occurs, the system of difference equations is more difficult to solve since \( T_n \) can no longer be ignored. Nevertheless, the heat flow is still a conserved quantity, and as long as the column can be operated at all, we can still introduce the plate efficiency independently of the plate number,

\[
y_n - y_{n+1} = P_n[y^n_0(T_n(y_{n+1}x_n(y_{n+1}))) - y_{n+1}] = P(q)f(y_{n+1}) \quad (23)
\]

i.e., we can, in principle, solve the system of equations for perfect plates first and afterward introduce the plate efficiency. Consequently, \( N(q) = N^*/p \), as before. If \( N^*(q) \) already exhibits local minima, then, of course, multiple solutions are possible. If \( N^*(q) \) decreases monotonically, as in the simpler case, then the arguments of section II show that multiple solutions are possible for sufficiently rapidly changing \( P(q) \).

(c) Imperfections of the column, e.g., heat leaks or pressure drops. If these imperfections can be incorporated into the plate efficiency, they obviously pose no problem since they do not change the general structure of the problem.

Of more interest is the fact that heat leaks can change the enthalpy flow along the column. This would result in a variation of the plate efficiency \( P(q) \) throughout the column. A similar variation of the plate efficiency can also occur; it is not a simple function of heat flow but instead of the material (liquid/vapor) flow, \( P_n = P(V_n) \). We recall that the liquid and vapor flows vary even for an otherwise perfect column and are only constant if \( \epsilon = 0 \) and \( C_{VL} = 0 \). Unless the effects of \( \epsilon, C_{VL} \neq 0 \) counteract each other to a certain degree justifying the approximation \( V_n = constant \). Therefore, the plate efficiencies will be different in different regions of the column, such that generally regions with large flows have a lower efficiency than those with smaller flows. The situation is turned around, if the minimum heat flow of the column is very small, because then an increase of \( q \) can actually improve the efficiency, since the stronger material flows will lead to milder frothing and thus increase the interaction between vapor and liquid. In either case, the last equality in eq 23 or eq 14 will no longer be true.

However, the formula \( N(q) = N^*/p \) will still hold in an average sense, i.e., by replacing \( P(q) \) by \( P_{average}(q) \). This is appropriate since the average plate efficiency of all the plates taken together will still decrease upon an increase of heat flow as long as the efficiency of the
individual plates decreases. If we assume that the plates are physically identical, then the variation of the heat/material flow along the column will result in a series of small drops in the average plate efficiency rather than one major change as the heat flow increases. Thus the change in the average plate efficiency will be spread out over a somewhat larger range of heat flow values compared to the change of an individual plate. Nevertheless, this effect is not very large, since, for example, the variation of vapor flow throughout the column typically is only a few percent of the total flow. In order to estimate this variation, we note from eqs 5–7 and the fact that usually $\epsilon \ll H$ that

$$V_{n+1} = \frac{Q}{H + \epsilon V_{n+1}} \approx \frac{Q}{H} \left[ 1 - \frac{\epsilon}{H} V_{n+1} + O\left(\frac{\epsilon^2}{H}\right) \right]$$  \hspace{1cm} \text{(24)}$$

Therefore the relative change of $V_n$ between feed and condenser is found to be small as long as $\epsilon \ll H$:

$$\frac{V_n - V_1}{V_1} = \frac{\left(1 - \frac{\epsilon}{H} x_r\right) - \left(1 - \frac{\epsilon}{H} x_D\right)}{1 - \frac{\epsilon}{H} x_D} \approx \frac{\epsilon}{H} (x_D - x_r) \ll 1$$  \hspace{1cm} \text{(25)}$$

Thus the general picture as described in section II still applies if we reinterpret $P(q)$ as the average plate efficiency during the process. A rapid change in the average plate efficiency causes an increase in the number of required plates, thus leading to the possibility of multiple operating modes.

Clearly, there are other imperfections and nonidealities we have not considered. Nevertheless, our results indicate that the existence of multiple operating modes for distillation columns should be quite common if the column is operated in a region of heat flows where the plate efficiency decreases rather rapidly.

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**Appendix**

Equation 14 is a differential equation that approximates the difference equation, eq 13. We will here justify that eq 14 generates solutions $y(n)$ which correspond to observed distillation profiles.

First we determine that $-d y / d n > 0$, i.e. the purity of component one increases toward the condenser as desired. We can write eq 14 as

$$-d y / d n = a/b$$

with

$$a = \alpha [Q y - x_D (e y + H)] - y b$$

$$b = Q - D (e y + H) + (\alpha - 1) [Q y - x_D (e y + H)]$$  \hspace{1cm} \text{(A1)}$$

We find that $b > 0$ since

$$Q - D (e y + H) = (V - D) (e y + H) > 0$$  \hspace{1cm} \text{(A2)}$$

and

$$Q y - x_D D (e y + H) = (y V - x_D D) (e y + H) > 0$$  \hspace{1cm} \text{(A3)}$$

This follows from eqs 5–7 and the fact that $\alpha > 1$, $H > 0$, and $\epsilon > -H$ for any ideal binary mixture. Similarly, since

$$\alpha = \frac{y^D(x)(1 - x)}{x[1 - y^D(x)]}$$  \hspace{1cm} \text{(A4)}$$

and

$$y V - x_D D = x L \quad \text{and} \quad y^D(x) > y$$  \hspace{1cm} \text{(A5)}$$

we find that

$$a = L (e y + H) \frac{(y^D(x) - y)(1 - x)}{1 - y^D(x)} > 0$$  \hspace{1cm} \text{(A6)}$$

Thus $-d y / d n > 0$, i.e. the purity of component one increases toward the condenser. We note that the value of heat flow that ensures that $V > D$ is given by $Q = q_D = D (x_D + H)$.

Next we determine the conditions under which $y(n)$ has an inflection point between the feed point and the condenser. This is a common feature of distillation processes and is indicative of the stability of the process with respect to perturbations in, for example, the heat flow.
flow/feed flow during the operation of the column. An inflection point is determined by \( d^2y/dn^2 = 0 \). Since \( dy/dn = 0 \), this is equivalent to \((d/dy)(dy/dn) = 0\). Solving the resulting quadratic equation, we find two possible inflection points:

\[
y^{\text{inf}} = \frac{(q_1 - q) \pm \sqrt{\alpha q(q - q_p)}}{(\alpha - 1)(q - q_p)}
\]

with \( q_0 \equiv \epsilon x_0 + H \), \( q_1 \equiv H(1 + (\alpha - 1)x_0) \), \( q_p \equiv \epsilon \frac{[1 + (\alpha - 1)x_0]}{(\alpha - 1)} \). \( \) (A7)

where \( q_p \begin{cases} > q_0 & \text{if } \epsilon > \epsilon_1 \\ < q_0 & \text{if } \epsilon < \epsilon_1 \end{cases} \) and \( \epsilon_1 \equiv H(\alpha - 1) \). \( \)

In all cases the minimum heat flow for achieving the desired purity is \( q_{\min} > q_p \), \( q_0 \). We note that \( y^{\text{inf}} \) are real when \( q > q_p \), but that one inflection point \( y^{\text{inf}} \) is always situated in the unphysical part of the plane. Furthermore, there is a simple pole at \( q = q_p \), which is only of relevance if \( \epsilon > \epsilon_1 \). The limiting values of \( y^{\text{inf}} \) are

\[
\lim_{q\to q_p} y^{\text{inf}}_+ = x_1 
\] (A8a)

with slope equal to \( \pm \infty \) depending on the value of \( \epsilon \) relative to \( \epsilon_1 \), while

\[
\lim_{q\to \infty} y^{\text{inf}}_+ = \frac{-1 \pm \sqrt{\alpha}}{\alpha - 1} 
\] (A8b)

with slope equal to \( 0^\pm \), depending on the values of \( \epsilon, \alpha \), and \( x_0 \).

In Figure 4 we plot \( y^{\text{inf}}_\pm(q) \) for \( \epsilon > \epsilon_1 \) (Figure 4a) and \( \epsilon < \epsilon_1 \) (Figure 4b). Note that, whereas for most choices of parameters, we find the inflection point \( y^{\text{inf}} \) in the interval \((0, x_0)\) in the physically interesting region \( q > q_{\min} \), combinations of parameters do exist where this is not the case. In addition, for any set of parameters there is always some interval of heat flows \( q \) for which both inflection points lie outside the physically relevant interval \((0, x_0)\). For reasons of space limitations we will not discuss the dependence of this interval on the parameters of the distillation process since it is quite complicated.

One can treat the stripping section of the distillation column analogously, and one finds that again two potential inflection points exist. The curves of \( y^{\text{inf}}_\pm(q) \) are qualitatively very similar to those presented in Figure 4, but the values of \( q_p, q_0, q_1 \), etc. are changed, of course.

**Literature Cited**


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