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Analysis of Combined Systems of Two Endoreversible Engines

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Abstract. A single endoreversible engine can operate as a cooler, a true heat engine, a heat pump, or a refrigerator. We investigate how many different modes of operation a combined system of two endoreversible engines may display. Special attention is paid to the independent combined system which neither consumes nor supplies power.

1. Introduction

Let us consider a Curzon-Ahlborn endoreversible engine [1]. The core is a reversible heat engine (e.g. Carnot), converting part of the heat flow Q into the work flow W. Further, we have two reservoirs at constant temperature: T_1 and T_2. Here $T_1 > T_2$. Finally, we have two linear conductances: g_1 and g_2. The points where the reversible engine is connected to the conductances are assumed to have temperatures T_3 and T_4, such that $T_1 - T_3 = Q/g_1$ and $T_4 - T_2 = (Q - W)/g_2$, see Fig. 1a.

We express that the inner part of the engine produces no entropy:

$$\frac{Q}{T_3} = \frac{Q-W}{T_4} \tag{1}$$

or

$$\frac{Q}{T_1 - Q/g_1} = \frac{Q-W}{T_2 + (Q-W)/g_2}\,, \tag{2}$$

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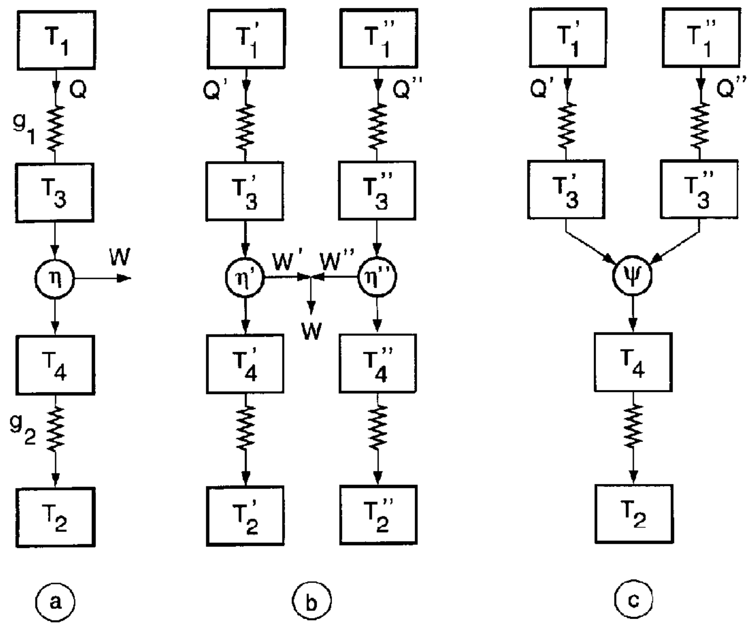

Fig. 1. Endoreversible engines: (a) endoreversible engine, (b) two combined endoreversible engines, (c) three-heat-reservoir engine.

where W is the produced power.

With the definition of η,

$$\eta \;=\; W/Q\,, \tag{3}$$

this yields

$$W(\eta) \;=\; \frac{g_1g_2}{g_1+g_2}\;\eta\;\frac{T_1-T_2-T_1\eta}{1-\eta}\,. \tag{4}$$

We are able to find the reversible value η_r of η by imposing $W=0$ (because no heat leaks are considered here). This yields, of course,

$$\eta_r \;=\; 1-T_2/T_1 \tag{5}$$

in accordance with Carnot's theorem. As a result we can rewrite (4) as

$$W(\eta) \;=\; c\eta\;\frac{\eta-\eta_r}{\eta-1}\,, \tag{6}$$

where c is a constant,

$$c \;=\; T_1\;\frac{g_1g_2}{g_1+g_2}\,, \tag{7}$$

see Fig. 2.

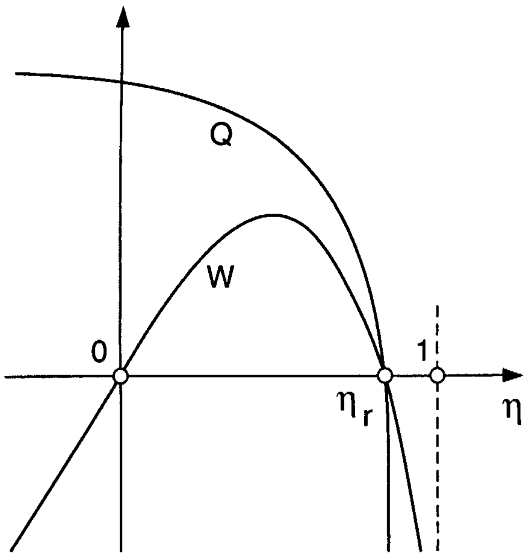

Fig. 2. Power/efficiency curves of an endoreversible engine.

It is interesting to note that the points $\eta = 0$ and $\eta = \eta_r$ divide the η-axis into three different parts [2, 3]:

(a) If $\eta < 0$, then the engine works as a cooler (speeding up an otherwise voluntary heat transfer).

(b) If $0 < \eta < \eta_r$, then the engine works as a true heat engine.

(c) If $\eta_r < \eta < 1$, then the engine works as a heat pump or a refrigerator (moving heat against a temperature gradient).

If we want to be complete, we have to consider also the case $\eta > 1$. However, the condition $\eta > 1$ necessarily involves either a negative T_3 or a negative T_4. Negative temperatures are significant for some physical systems (e.g. lasers), but the possibility of a reversible engine working between two reservoirs of opposite temperature sign is a matter of intense debate [4 – 7]. We therefore believe such a Carnot engine working between negative T_3 and positive T_4, or vice versa, can be omitted in the framework of energy physics and engineering. In the below analyses of the present paper only the three cases with $\eta < 1$ will be considered.

2. Two Endoreversible Engines

Now we join two such Curzon-Ahlborn engines together: one with single prime: T_1', T_2', T_3', T_4', and η' and one with double prime: T_1'', T_2'', T_3'', T_4'', and η''. We denote by W the total delivered power $W' + W''$ (see Fig. 1b). Without loss of generality, we can attribute primes in such a way that $\eta'_r \leq \eta''_r$.

2.1. The general case

We now have two degrees of freedom: η' and η''. How then is the (η', η'')-plane divided into different parts?

As explained above, each endoreversible engine has three regions on the η-axis. Therefore, one expects at first sight that two endoreversible engines should have 3×3=9 regions in the (η', η'')-plane, separated by six simple straight lines $\eta' = 0$, $\eta' = \eta'_r$, $\eta' = 1$, $\eta'' = 0$, $\eta'' = \eta''_r$, and $\eta'' = 1$.

However, as we shall see below, the region

$$\eta'_r < \eta' < 1 \quad \text{and} \quad 0 < \eta'' < \eta''_r$$

will be divided further into two subregions by the line $\eta'' = \eta'$, resulting in a total of 10 regions (see Fig. 3a).

First we consider three (rather trivial) rectangular regions (denoted A through C):

A. In the region

$$0 < \eta' < \eta'_r \quad \text{and} \quad 0 < \eta'' < \eta''_r$$

two heat engines are working together for the production of power W. Uncorrelated choices for T'_1, T'_2, T''_1 and T''_2 are of little practical significance. More interesting situations occur when the two engines operate either in parallel (e.g. $T''_2 = T'_2$) or in series (e.g. $T''_2 = T'_1$), see references [8] and [9].

B. In the region

$$\eta' < 0 \quad \text{and} \quad \eta'' < 0$$

two coolers are powered independently. Again, these two devices can be powered either in parallel or in series.

C. In the region

$$\eta'_r < \eta' < 1 \quad \text{and} \quad \eta''_r < \eta'' < 1$$

two heat pumps or refrigerators are powered independently (either in parallel or series). The case where two refrigerators are combined in series was discussed in reference [10].

Then we have four more simple rectangular regions (D through G):

D. In the region

$$\eta'_r < \eta' < 1 \quad \text{and} \quad \eta'' < 0$$

one heat pump or refrigerator ($'$) and one cooler ($''$) are operating.

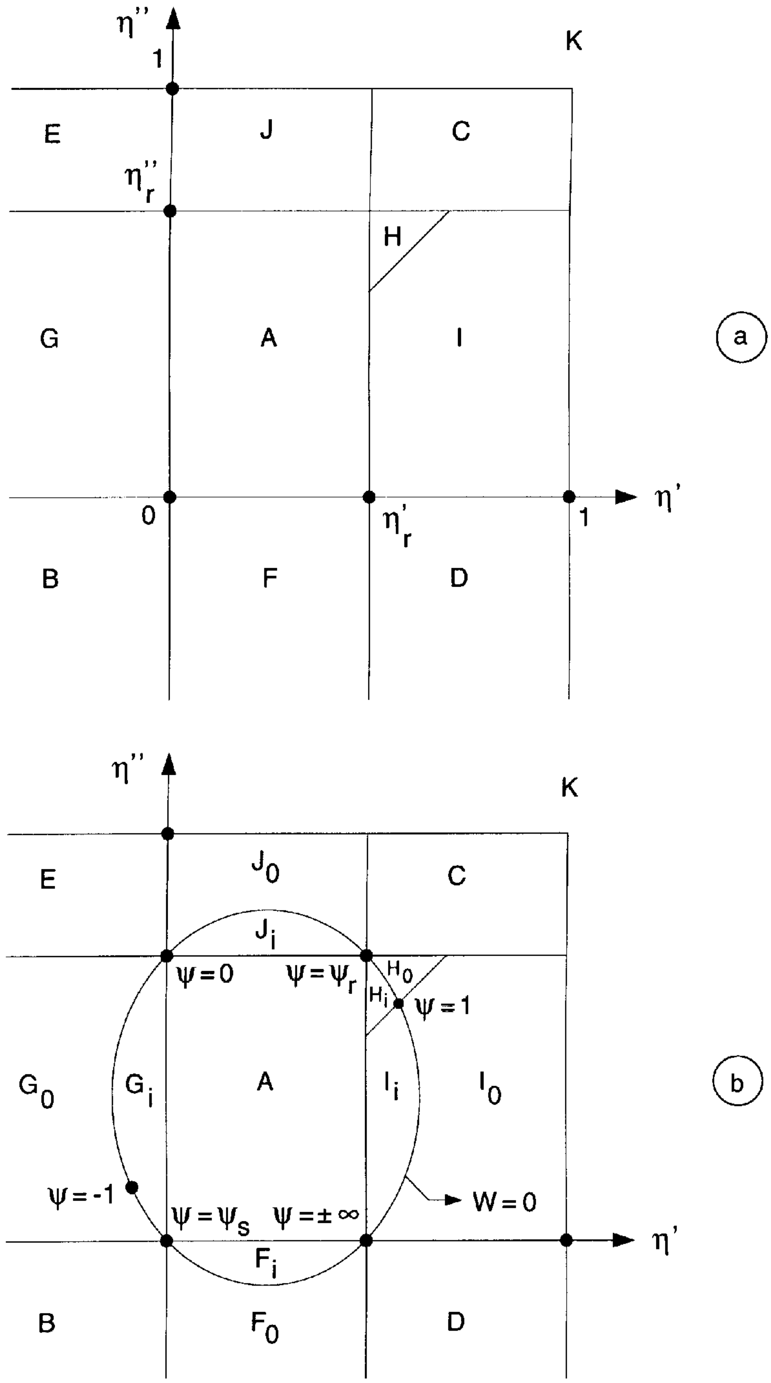

Fig. 3. Division of the (η', η'')-plane for two coupled endoreversible engines: (a) basic division due to character of operation, (b) curve of no net work exchange ($W = 0$).

E. In the region

$$\eta' < 0 \quad \text{and} \quad \eta''_r < \eta'' < 1$$

one heat pump or refrigerator ($''$) and one cooler ($'$) are operating.

F. In the region

$$0 < \eta' < \eta'_r \quad \text{and} \quad \eta'' < 0$$

one endoreversible engine ($'$) works as a heat engine and the other engine ($''$) works as a cooler.

G. In the region

$$\eta' < 0 \quad \text{and} \quad 0 < \eta'' < \eta''_r$$

one endoreversible engine ($''$) works as a heat engine and the other one ($'$) as a cooler.

Further we have one triangular region (H) and one pentagonal region (I):

H. In the region

$$\eta'_r < \eta' < 1\,, \qquad 0 < \eta'' < \eta''_r \quad \text{and} \quad \eta'' > \eta'$$

the more efficient engine ($''$) works as a heat engine and the other engine ($'$) as a heat pump.

- When $T''_2 = T'_1$, the combined system may be called 'the first type heat pump' or 'true heat pump' because the total heating $(1-\eta'')Q'' - Q'$ of the reservoir at temperature T''_2 is larger than Q'' alone, see Figure 1b/ of reference [11]. We may thus speak of a 'heating booster'. It should be pointed out that the refrigerators in Figure 1 of reference [12] and Figure 1 of reference [13] also operate in this region.
- When $T''_1 = T'_2$, the combined system may be called 'the second type heat pump' or 'heat transformer' because the total heating $-Q'$ of the reservoir at temperature T'_1 is smaller than the heat $-(1-\eta')Q' + Q''$ withdrawn from the reservoir at temperature T'_2 but at a higher temperature than the source at T''_1, see reference [14]. We may speak of a 'temperature booster', since the high temperature T''_1 is used to realise an even higher temperature T'_1, see Figure 1b of reference [11].
- When $T''_2 = T'_2$, the combined system may be conceived as a heat pump or a refrigerator.
- Finally, when $T''_1 = T'_1$, the combined system may be conceived as a heat transformer.

I. In the region

$$\eta_r' \;<\; \eta' \;<\; 1\,, \qquad 0 \;<\; \eta'' \;<\; \eta_r'' \quad \text{and} \quad \eta'' \;<\; \eta'$$

the less efficient engine ($''$) works as a heat engine and the other engine ($'$) as a heat pump.

- When $T_2'' = T_1'$ or $T_1'' = T_2'$, the combined systems are similar to those in region H [11 – 14].
- When $T_2'' = T_2'$ or $T_1'' = T_1'$, the combined systems are meaningless because the same heating load may be realised more efficiently by direct conduction.

Thus there exists an obvious difference between regions H and I.

J. In region

$$0 \;<\; \eta' \;<\; \eta_r' \quad \text{and} \quad \eta_r'' \;<\; \eta'' \;<\; 1$$

one engine ($'$) works as a heat engine and the other ($''$) as a heat pump or a refrigerator.

- When $T_2' = T_1''$ or $T_1' = T_2''$, the combined systems are similar to case H.
- When $T_2' = T_2''$, the combined system may be conceived as a heat transformer.
- When $T_1' = T_1''$, the combined system may be conceived as a heat pump or a refrigerator.

Finally, the rest of the (η', η'')-plane, i.e.

K. The region

$$\eta' \;>\; 1 \quad \text{or} \quad \eta'' \;>\; 1$$

is the large eleventh region with negative temperatures, which is unreal in engineering, as explained above in the Introduction.

2.2. The special case $W = 0$

In general, W is not equal to 0 for the combined system of two endoreversible engines, so that they form together no selfcontained system. The case $W = 0$ is, however, a very interesting one, as it represents an independent system, neither consuming nor producing power. In other words: the power produced by one endoreversible engine is consumed by the other endoreversible engine. We then have only one degree of freedom, as η' and η'' are correlated by the condition

$$W(\eta', \eta'') \;=\; 0\,, \tag{8}$$

i.e.

$$c'\eta'(\eta' - \eta_r')(\eta'' - 1) + c''\eta''(\eta'' - \eta_r'')(\eta' - 1) \; = \; 0\,. \tag{9}$$

Thus we are on a third-degree curve in the $(\eta',\ \eta'')$-plane. This polynomial curve passes through the four points $\{0,\ 0\}$, $\{0,\ \eta_r''\}$, $\{\eta_r',\ 0\}$, and $\{\eta_r',\ \eta_r''\}$. It also passes through the five regions F, G, H, I, and J, subdividing these into ten subregions, five of which (F_i, G_i, H_i, I_i, and J_i) lie inside the closed polynomial curve (producing net power: $W > 0$), and five of which (F_o, G_o, H_o, I_o, and J_o) lie outside the curve (thus consuming net power: $W < 0$), see Fig. 3b.

The combined systems operating in the 6 regions inside the curve have power output. The combined systems operating in the 9 regions (the K-region is not considered) outside the curve need power input. The combined systems operating on the curve are the independent systems which have only one degree of freedom.

Which parameter should be chosen as the single degree of freedom? We have two equivalent choices:

$$\psi \; = \; \eta'/\eta'' \; = \; (W'/Q')/(W''/Q'') \; = \; -Q''/Q'\,, \tag{10}$$

$$\phi \; = \; \eta''/\eta' \; = \; (W''/Q'')/(W'/Q') \; = \; -Q'/Q''\,. \tag{11}$$

For convenience we choose ψ. The usefulness of this choice can be demonstrated by e.g. calculating Q' and Q'', the amounts of pumped heat. As (see (6) and (7))

$$Q' \; = \; c'\,\frac{\eta' - \eta_r'}{\eta' - 1}\,, \tag{12}$$

we can eliminate η' and η'' from the set (9), (10), (12) and obtain an (implicit) expression for $Q'(\psi)$:

$$\psi(\psi - 1)Q'^2 - (c'\psi^2 - c''\eta_r''\psi - c'\eta_r'\psi + c'')Q' - c'c''(\eta_r''\psi - \eta_r') \; = \; 0\,. \tag{13}$$

Analogously, we may obtain a quadratic equation for Q'', but it is advantageous to make use of

$$Q''(\psi) \; = \; -\psi\, Q'(\psi)\,, \tag{14}$$

once $Q'(\psi)$ is constructed from (13), see Fig. 4.

On the curve $W(\eta',\eta'') = 0$ in Fig. 3b, the variable ψ plays the role of a parameter which evolves along the curve. The parameter representation $\{\eta'(\psi),\ \eta''(\psi)\}$ of the curve can be constructed. This only requires the solution of a quadratic equation.

Note that we have one reversible value of ψ:

$$\psi_r \; = \; \eta_r'/\eta_r''\,, \tag{15}$$

for which $\eta' = \eta_r'$ and $\eta'' = \eta_r''$. For $\psi = -(c''/c')(1/\psi_r)$ (for short: $\psi = \psi_s$) we have $\eta' = \eta'' = 0$. We finally remark that, in contrast to η' and η'', the quantity

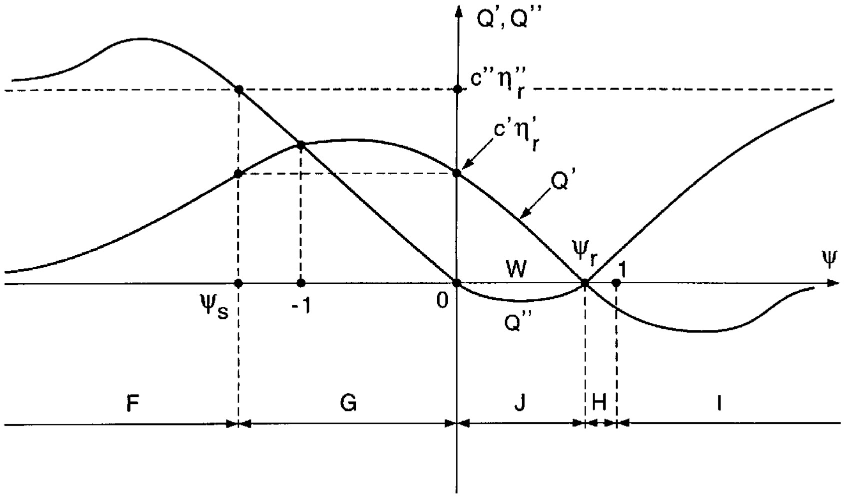

Fig. 4. Power/efficiency curves of two combined endoreversible engines.

ψ can take any value from $-\infty$ to $+\infty$, and thus may be larger than unity. Thus the ψ-axis of the combined endoreversible engines is different from the η-axes of the separate endoreversible engines.

The ψ-axis is divided into 5 parts. We shortly recall here the different modes of operation:

(F) and (G): If $\psi < 0$, then one engine works as a true heat engine and the other as a cooler.

(H), (I) and (J): If $0 < \psi$, then one engine works as a true heat engine and the other as a refrigerator. The combined system may work as a true heat pump, a refrigerator, or a heat transformer, depending on their relative parameters.

The symmetric results, i.e. the results in terms of ϕ, can be easily constructed.

3. Alternative Heat Engine

Next we consider the engine of references [11], [12] and [14], which dispenses with work exchange all together. It has not four, but only three external temperatures: T_2, T_1', and T_1'' and thus may be viewed either as a compact version of the combined endoreversible engines (Fig. 1b) or a generalized version of a single endoreversible engine (Fig. 1a) with a heat reservoir instead of the work reservoir. Analogously there are not four, but only three intermediate temperatures: T_4, T_3', and T_3'', see Fig. 1c. The scheme is similar to the 'tri-cycle' engine proposed by Andresen et al. [15]. Without loss of generality we can attribute primes such that $T_1' \leq T_1''$.

This thermal engine has only one degree of freedom, ζ, which we define as

$$\zeta \;=\; \frac{(T_3' - T_4)T_3''}{T_3'(T_3'' - T_4)}\,. \tag{16}$$

We express conservation of entropy in the reversible 'core' as

$$\frac{Q'}{T_3'} + \frac{Q''}{T_3''} \;=\; \frac{Q' + Q''}{T_4}\,. \tag{17}$$

From these two equations it follows immediately that

$$Q'' \;=\; -\zeta Q'\,. \tag{18}$$

Eliminating Q'' from this result and from

$$\frac{Q'}{T_1' - Q'/g_1'} + \frac{Q''}{T_1'' - Q''/g_1''} \;=\; \frac{Q' + Q''}{T_2 + (Q' + Q'')/g_2} \tag{19}$$

yields:

$$\zeta(\zeta - 1)Q'^2 - d\,(d'\zeta^2 - f\zeta + d'')Q' - d\,[\; T_1'(T_1'' - T_2)\zeta - T_1''(T_1' - T_2)\;] \;=\; 0\,, \tag{20}$$

where

$$d \;=\; g_1'g_1''g_2/(g_1' + g_1'' + g_2)\,, \tag{21}$$
$$d' \;=\; T_1'(1/g_1'' + 1/g_2)\,, \tag{22}$$
$$d'' \;=\; T_1''(1/g_1' + 1/g_2) \quad \text{and} \tag{23}$$
$$f \;=\; (T_1' - T_2)/g_1'' + (T_1'' - T_2)/g_1' + (T_1'' + T_1')/g_2\;. \tag{24}$$

Analogously, we can eliminate Q' in order to find an equation for Q''. Both Q' and Q'' are zero for ζ equal to its reversible value,

$$\zeta_r \;=\; \frac{(T_1' - T_2)T_1''}{T_1'(T_1'' - T_2)}\,. \tag{25}$$

Since $T_1' \leq T_1''$, we have $\zeta_r \leq 1$.

Equation (20) is not identical to, but nevertheless very similar to equation (13). Therefore, a graph representing curves $Q'(\zeta)$ and $Q''(\zeta)$ deduced from equations (18) and (20) is quite similar to Fig. 4.

4. Conclusion

Whereas the single endoreversible engine has three modes of operation (cooler, engine, pump), we find ten modes of operation for the combination of two endoreversible engines. In the special case of a closed combination, exchanging no power with the external world, five modes may be distinguished.

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