Solutions to the problems in Chapter 17

17.4 or 17.5 AC conductivity

In the presence of a time-dependent (uniform) electrical field

$$\vec{E} = \vec{E}_0 e^{-i\omega t} \tag{1}$$

we may use the general solution of the Boltzmann equation given by (17.24), before the integration with respect to t' is performed:

$$g = f - \int_{-\infty}^{t} dt'\, e^{(t'-t)/\tau_\varepsilon} e^{-i\omega t'} \vec{v}_{\vec{k}} \cdot \vec{E}_0\, e\, \frac{\partial f(t')}{\partial \mu} \tag{2}$$

Including only effects which are linear in the applied field – the linearized Boltzmann equation – then $\vec{v}_{\vec{k}}(t')$ and $f(t')$ may be replaced by their time-independent equilibrium values at zero field (the time-dependent changes of these quantities are $\propto E$). The integration is then performed straightforwardly, and

$$g = f - \frac{\tau_\varepsilon}{1 - i\omega\tau_\varepsilon} \vec{v}_{\vec{k}} \cdot \vec{E}_0\, e\, \frac{\partial f}{\partial \mu} e^{-i\omega t} = f - \frac{\tau_\varepsilon}{1 - i\omega\tau_\varepsilon} \vec{v}_{\vec{k}} \cdot \vec{E}\, e\, \frac{\partial f}{\partial \mu} \tag{3}$$

Introducing this expression into (17.43)-(17.44) we get the frequency-dependent conductivity (valid in the limit of $\vec{E}_0 \to \vec{0}$)

$$\sigma_{\alpha\beta}(\omega) = e^2 \int [d\vec{k}] \frac{\tau_\varepsilon}{1 - i\omega\tau_\varepsilon} v_\alpha v_\beta \frac{\partial f}{\partial \mu} \tag{4}$$

or in the case of a cubic or an isotropic (free electron) system:

$$\sigma(\omega) = \frac{ne^2}{m^*} \frac{\tau}{1 - i\omega\tau} = \frac{ne^2\tau}{m^*} \frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \tag{5}$$

17.5 or 17.6 Current driven by thermal gradient

We shall consider a metal subject to a constant temperature gradient

$$\nabla T = \left(\frac{\partial T}{\partial x}, 0, 0 \right) \qquad \text{and} \qquad \varepsilon_{\vec{k}} = \varepsilon_k = \tfrac{1}{2} m^* \vec{v}_{\vec{k}}^{\,2} \tag{1}$$

According to (17.60), (17.62), and (17.68) the electrical current, in the case of $G = 0$, is

$$\vec{j} = \mathbf{L}^{12} \left(-\frac{\nabla T}{T} \right), \quad \mathbf{L}^{12} = -\frac{1}{e} \mathcal{L}^{(1)} = -\frac{1}{e} \frac{\pi^2}{3} (k_\mathrm{B} T)^2 \overline{\overline{\sigma}}\,'(\varepsilon_\mathrm{F}) \tag{2}$$

The assumption of an isotropic mass, (1), implies $\overline{\overline{\sigma}}(\varepsilon)$ to be diagonal, and according to (17.64) the diagonal element is

$$\sigma_{\alpha\alpha}(\varepsilon) = e^2 \tau \int d\vec{k} D_{\vec{k}}\, v_{\vec{k}\alpha}^2\, \delta(\varepsilon - \varepsilon_{\vec{k}}) = e^2 \tau \int d\varepsilon_k D(\varepsilon_k)\, \tfrac{1}{3} \vec{v}_{\vec{k}}^{\,2}\, \delta(\varepsilon - \varepsilon_k) \tag{3}$$

The integration of $v_{\vec{k}\alpha}^2$ over all solid angles, at a constant $|\vec{k}|$, is 1/3 of the result deriving from $\mathrm{Tr}\, v_{\vec{k}\alpha}^2 = \vec{v}_{\vec{k}}^{\,2}$ and using $\vec{v}_{\vec{k}}^{\,2} = 2\varepsilon_k / m^*$, we get

$$\sigma_{\alpha\alpha}(\varepsilon) = e^2 \tau D(\varepsilon) \frac{2\varepsilon}{3m^*} \quad \Rightarrow \quad \sigma'_{\alpha\alpha}(\varepsilon) = \frac{2e^2\tau}{3m^*} \left[D(\varepsilon) + \varepsilon\, D'(\varepsilon) \right] \tag{4}$$

This result is introduced in (2)

$$\vec{j} = \frac{1}{e}\frac{\pi^2}{3}(k_\mathrm{B}T)^2\frac{2e^2\tau}{3m^*}D(\varepsilon_\mathrm{F})\left(1+\varepsilon_\mathrm{F}\frac{D'(\varepsilon_\mathrm{F})}{D(\varepsilon_\mathrm{F})}\right)\frac{\nabla T}{T} \tag{5}$$

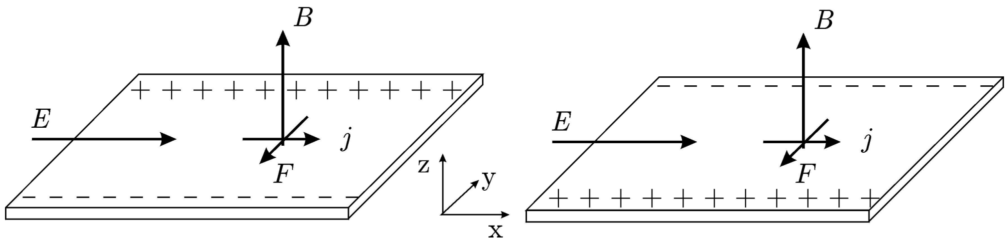
In terms of the heat capacity $c_V = (\pi^2/3)k_\mathrm{B}^2 T D(\varepsilon_\mathrm{F})$, (6.77), we finally get

$$j_x = \underline{\frac{2e\tau c_V}{3m^*}\frac{\partial T}{\partial x}\left(1+\varepsilon_\mathrm{F}\frac{D'(\varepsilon_\mathrm{F})}{D(\varepsilon_\mathrm{F})}\right)} = \frac{e\tau c_V}{m^*}\frac{\partial T}{\partial x} \tag{6}$$

where the last equality sign is valid only if $D(\varepsilon) \propto \sqrt{\varepsilon}$.

17.8 or 17.9 Hall effect – elementary argument

a) Electrons b) Holes



The Hall effect geometry: The applied field $\vec{E}$ is along the x axis leading to a current $\vec{j}$ in this direction, i.e. an electron (hole) current in the minus (plus) x direction. The magnetic part of the Lorenz force $\vec{F} \propto \vec{j} \times \vec{B}$ is in the minus y direction, when $\vec{B}$ is along z, leading to opposite signs of the resulting charge distributions in the electron and hole cases.

(a) The equation of motion, when assuming the Drude model, is

$$m\dot{\vec{v}} = -e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) - \frac{m\vec{v}}{\tau} \tag{1}$$

in the case of electrons with charge $-e$. Using $\vec{B} = (0,0,B)$, then we get $\vec{v}\times\vec{B} = (v_y B, -v_x B, 0)$ and since the current, by geometry, is constrained to be along x, the steady state is characterized by v_x being constant and $v_y = 0$. These conditions imply

$$m\dot{v}_y = -e\left(E_y - \frac{v_x}{c}B\right) - 0 = 0 \quad\Rightarrow\quad E_y = \frac{v_x}{c}B \tag{2}$$

(b) The current is $\vec{j} = (j_x, 0, 0)$ with $j_x = -nev_x$, and the Hall coefficient is

$$\mathcal{R} = \frac{E_y}{B\,j_x} = \frac{v_x}{c}B\frac{1}{B(-nev_x)} = \underline{-\frac{1}{nec}}\,, \qquad E_y = \mathcal{R}B\,j_x \tag{3}$$

The electric field in the x direction may be determined from $\dot{v}_x = 0$, or

$$m\dot{v}_x = -eE_x - \frac{mv_x}{\tau} = 0 \quad\Rightarrow\quad E_x = -\frac{me}{\tau}v_x = -\frac{me}{\tau}\frac{j_x}{(-ne)} = \frac{m\,j_x}{ne^2\tau} = \frac{j_x}{\sigma} \tag{4}$$

i.e. E_x is determined by the Drude resistivity σ^{-1} as in the case of $B = 0$.

17.9 or 17.10 Hall effect – Boltzmann equation

The Boltzmann equation in the relaxation-time approximation is given by (17.17) and (17.18)

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \dot{\vec{r}} \cdot \frac{\partial g}{\partial \vec{r}} + \dot{\vec{k}} \cdot \frac{\partial g}{\partial \vec{k}} = -\frac{g-f}{\tau} \tag{1}$$

When the state is uniform in space and steady in time, this equation reduces to

$$\dot{\vec{k}} \cdot \frac{\partial g}{\partial \vec{k}} = -\frac{g-f}{\tau} \tag{2}$$

The semiclassical equation of motion is

$$\hbar\dot{\vec{k}} = -e\left(\vec{E} + \frac{1}{c}\vec{v}_{\vec{k}} \times \vec{B}\right), \qquad \vec{v}_{\vec{k}} = \dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} = \frac{\hbar\vec{k}}{m^*} \tag{3}$$

where the second equation expresses that the mass tensor is assumed to be isotropic for simplicity. Like in Problem 17.4 we are only interested in the response (current) which is linear in the electric field. This means that g may be replaced by f in products on the left hand side of (2) which already involve $\vec{E}$. The linearized version of the Boltzmann equation (2) is therefore

$$-\frac{e}{\hbar}\vec{E} \cdot \frac{\partial f}{\partial \vec{k}} - \frac{e}{\hbar c}\vec{v}_{\vec{k}} \times \vec{B} \cdot \frac{\partial (g-f)}{\partial \vec{k}} = -\frac{g-f}{\tau} \tag{4}$$

when using that $\vec{v}_{\vec{k}} \times \vec{B} \cdot \dfrac{\partial f}{\partial \vec{k}} = 0$ because $\dfrac{\partial f}{\partial \vec{k}}$ is parallel with $\vec{v}_{\vec{k}}$, as

$$\frac{\partial f}{\partial \vec{k}} = \frac{\partial f}{\partial \varepsilon_{\vec{k}}}\frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}} = -\frac{\partial f}{\partial \mu}\hbar\,\vec{v}_{\vec{k}} \tag{5}$$

(a) The geometry is the same as applied in problem 17.8, hence we define $\vec{B} = (0,0,B)$ and assume the resulting $\vec{E} = (E_x, E_y, 0)$ to be perpendicular to $\vec{B}$. In this geometry, we guess that the solution has the form

$$g = f + a\,k_x + b\,k_y \tag{6}$$

Introducing this in eq. (4) and utilizing (5), we get

$$\frac{e\hbar}{m^*}\frac{\partial f}{\partial \mu}\left(E_x\,k_x + E_y\,k_y\right) - \frac{eB}{m^* c}\left(a\,k_y - b\,k_x\right) = -\frac{a\,k_x + b\,k_y}{\tau} \tag{7}$$

Since k_x and k_y are independent variables, this equation leads to two independent conditions, which determine a and b to be

$$a = \frac{E_x - \omega_c\tau E_y}{1+(\omega_c\tau)^2}\left(-\frac{e\hbar\tau}{m^*}\right)\frac{\partial f}{\partial \mu}, \qquad b = \frac{E_y + \omega_c\tau E_x}{1+(\omega_c\tau)^2}\left(-\frac{e\hbar\tau}{m^*}\right)\frac{\partial f}{\partial \mu} \tag{8}$$

where we have introduced the cyclotron frequency $\omega_c = \dfrac{eB}{m^* c}$.

The α component of the current density $\vec{j}$ is, according to Marder's eq. (17.43),

$$\begin{aligned} j_\alpha &= -e\int [d\vec{k}]\, v_{\vec{k}\alpha}(g_{\vec{k}} - f_{\vec{k}}) = -e\int [d\vec{k}]\, v_{\vec{k}\alpha}\left(a\,k_x + b\,k_y\right) \\ &= -\frac{em^*}{\hbar}\int [d\vec{k}]\left(a\,v_{\vec{k}x}^2\delta_{\alpha x} + b\,v_{\vec{k}y}^2\delta_{\alpha y}\right) \end{aligned} \tag{9}$$

where the last equality sign follows because the off-diagonal terms vanish, when the mass tensor is assumed to be diagonal (isotropic). Using the same procedure as in Marder's eqs. (17.44)-(17.50), we have

$$-\frac{em^*}{\hbar}\int [d\vec{k}]\, v_{\vec{k}\alpha}^2 \left(-\frac{e\hbar\tau}{m^*}\right)\frac{\partial f}{\partial\mu} = e^2\tau \int [d\vec{k}]\, v_{\vec{k}\alpha}^2 \frac{\partial f}{\partial\mu} = \frac{ne^2\tau}{m^*} \equiv \sigma_0 \qquad (10)$$

and combining the three equations (8)-(10), we finally get

$$j_x = \frac{E_x - \omega_c\tau E_y}{1+(\omega_c\tau)^2}\,\sigma_0\,, \qquad j_y = \frac{E_y + \omega_c\tau E_x}{1+(\omega_c\tau)^2}\,\sigma_0 \qquad (11)$$

In the case where $\vec{E}$ is assumed to be along the z axis parallel to the field, we have to add a term $c_z\,k_z$ to the trial function g in (6). In this situation, the magnetic field does not contribute to the Boltzmann equation, and we get $j_z = \sigma_0 E_z$. Hence, for a system with an isotropic mass m^*, the total conductivity tensor is found to be

$$\overline{\overline{\sigma}} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & 1+(\omega_c\tau)^2 \end{pmatrix}\,, \qquad \omega_c = \frac{eB}{m^*c} \qquad (12)$$

when the magnetic field B is applied along the z axis.

This result may also be expressed in terms of the Hall coefficient $\mathcal{R}$, where

$$\omega_c\tau = -\mathcal{R}B\sigma_0, \qquad \text{or} \qquad \mathcal{R} = -\frac{1}{nec} \qquad (13)$$

The resistivity tensor $\overline{\overline{\rho}}$, defined by the relation $\vec{E} = \overline{\overline{\rho}}\,\vec{j}$, is the inverse of the conductivity tensor and is particularly simple

$$\overline{\overline{\rho}} = \overline{\overline{\sigma}}^{\,-1} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau & 0 \\ -\omega_c\tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & -\sigma_0\mathcal{R}B & 0 \\ \sigma_0\mathcal{R}B & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (14)$$

which is in perfect agreement with the results derived from the Drude model (problem 17.8). The calculations in section 17.4.8 assume $\omega_c\tau \gg 1$, in which case the diagonal elements of $\overline{\overline{\sigma}}$ may be neglected in comparison with the off-diagonal ones, $\sigma_{xx} = \sigma_{yy} \simeq 0$ and $\sigma_{xy} = -\sigma_{yx} \simeq -(\omega_c\tau)^{-1}$.

Solution to HS's problem 3

Hall effect of a two-dimensional electron gas

A two-dimensional electron gas with an anisotropic dispersion

$$\varepsilon = a(k_x^2 + k_y^2) + b(k_x^4 + k_y^4), \qquad a > 0, \quad b > 0 \tag{1}$$

Introducing the polar angle θ in the (k_x, k_y)-coordinate system of the reciprocal lattice, the dispersion relation may be written

$$\varepsilon = a\,k^2 + \tfrac{1}{4}b\,(3 + \cos 4\theta)k^4, \qquad k_x = k\cos\theta, \quad k_y = k\sin\theta \tag{2}$$

reflecting directly the four-fold, cubic symmetry of the dispersion.

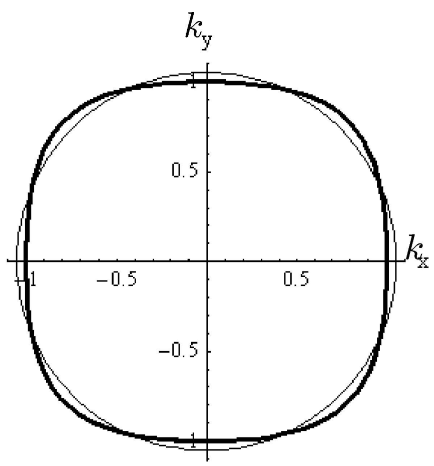
1) The equation determining the constant energy contour is obtained by solving (2) with respect to k^2

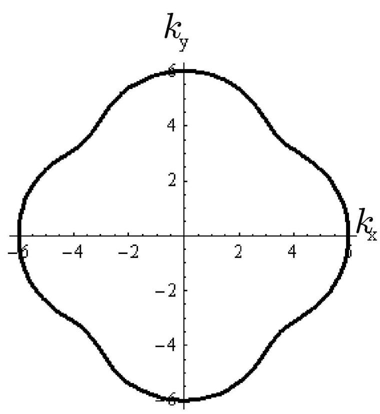
$$k^2 = k^2(\varepsilon) = \frac{2a}{b(3+\cos 4\theta)}\left[\left(1 + \frac{b\varepsilon}{a^2}(3+\cos 4\theta)\right)^{1/2} - 1\right] \tag{3}$$

In the case of $b\varepsilon \ll a^2$, the square root may be expanded, and to second order ($\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2$) the result is

$$k^2(\varepsilon) \simeq \frac{\varepsilon}{a}\left(1 - \frac{3b\varepsilon}{4a^2}\right) - \frac{b\varepsilon^2}{4a^3}\cos 4\theta \tag{4}$$

In the figure below (to the left) I show a constant energy contour, which differs visible from a circle. It is obtained by a numerical evaluation of (3) (Mathematica program) in the case of $a = b = 1$ and $\varepsilon = 2$ (assuming dimensionless quantities). The thin line shows the average length of $\vec{k}(\varepsilon)$. The figure to the right show the corresponding $|\nabla_{\vec{k}}\varepsilon_{\vec{k}}|$ as a function of the angle θ. The gradient, and hence the velocity $\vec{v}_{\vec{k}} = \nabla_{\vec{k}}\varepsilon_{\vec{k}}/\hbar$, is perpendicular to the constant energy contour. Notice that $|\vec{v}_{\vec{k}}|$ is smallest along the $\langle 11\rangle$ directions, where $|\vec{k}(\varepsilon)|$ has its maxima.





2) The Boltzmann equation in the relaxation-time approximation, (17.17)-(17.18),

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \dot{\vec{r}}\cdot\frac{\partial g}{\partial \vec{r}} + \dot{\vec{k}}\cdot\frac{\partial g}{\partial \vec{k}} = -\frac{g-f}{\tau} \quad \Rightarrow \quad \dot{\vec{k}}\cdot\frac{\partial g}{\partial \vec{k}} = -\frac{g-f}{\tau} \tag{5}$$

when considering the steady state of a uniform system. The fields are assumed to be $\vec{E} = (E, 0, 0)$ and $\vec{B} = (0, 0, B)$, and the semiclassical equation of motion is

$$\hbar\dot{\vec{k}} = -e\left(\vec{E} + \frac{1}{c}\vec{v}_{\vec{k}} \times \vec{B}\right), \qquad \vec{v}_{\vec{k}} = \dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial\varepsilon_{\vec{k}}}{\partial\vec{k}} \tag{6}$$

Introducing this in the right-hand part of (5) we get

$$-\frac{e}{\hbar}\left(\vec{E} + \frac{1}{c}\vec{v}_{\vec{k}} \times \vec{B}\right) \cdot \left(\frac{\partial(g-f)}{\partial\vec{k}} + \frac{\partial f}{\partial\vec{k}}\right) = -\frac{g-f}{\tau} \tag{7}$$

where

$$\frac{\partial f}{\partial\vec{k}} = \frac{\partial f}{\partial\varepsilon_{\vec{k}}}\frac{\partial\varepsilon_{\vec{k}}}{\partial\vec{k}} = \hbar\vec{v}_{\vec{k}}\frac{\partial f}{\partial\varepsilon_{\vec{k}}} \tag{8}$$

This gradient is perpendicular to $\vec{v}_{\vec{k}} \times \vec{B}$, and the linearized version of (7) is

$$-e\vec{E} \cdot \vec{v}_{\vec{k}}\frac{\partial f}{\partial\varepsilon_{\vec{k}}} - \frac{e}{\hbar c}\vec{v}_{\vec{k}} \times \vec{B} \cdot \frac{\partial(g-f)}{\partial\vec{k}} = -\frac{g-f}{\tau} \tag{9}$$

Since $(g-f)$ is going to scale with E, the term neglected, $-\frac{e}{\hbar}\vec{E} \cdot \frac{\partial(g-f)}{\partial\vec{k}}$, is of second order in E. Inserting $\vec{E} = (E, 0, 0)$ and $\vec{B} = (0, 0, B)$ in (9), we finally get

$$-ev_x E\frac{\partial f}{\partial\varepsilon_{\vec{k}}} = \left[\frac{eB}{\hbar c}\left(v_y\frac{\partial}{\partial k_x} - v_x\frac{\partial}{\partial k_y}\right) - \frac{1}{\tau}\right](g-f) \tag{10}$$

3) With the assumption of $g - f = k_y F(\varepsilon)$, we find

$$\begin{aligned} v_y\frac{\partial(g-f)}{\partial k_x} - v_x\frac{\partial(g-f)}{\partial k_y} &= v_y k_y\frac{dF}{d\varepsilon}\frac{\partial\varepsilon}{\partial k_x} - v_x F - v_x k_y\frac{dF}{d\varepsilon}\frac{\partial\varepsilon}{\partial k_y} \\ &= v_y k_y\frac{dF}{d\varepsilon}\hbar v_x - v_x F - v_x k_y\frac{dF}{d\varepsilon}\hbar v_y = -v_x F \end{aligned} \tag{11}$$

When introducing this result in (10) and neglecting $1/\tau$, we get

$$-ev_x E\frac{\partial f}{\partial\varepsilon_{\vec{k}}} = \frac{eB}{\hbar c}\left[-v_x F(\varepsilon)\right] \quad \Rightarrow \quad F(\varepsilon) = \frac{\hbar c E}{B}\frac{\partial f}{\partial\varepsilon_{\vec{k}}} \tag{12}$$

The current in the y direction is then

$$\begin{aligned} j_y &= -e\int[d\vec{k}]v_y(g-f) = -\frac{e\hbar cE}{B}\int[d\vec{k}]v_y k_y\frac{\partial f}{\partial\varepsilon_{\vec{k}}} \\ &= -\frac{e\hbar cE}{B}\int dk_x\int dk_y\, D_{\vec{k}}\frac{\partial\varepsilon_{\vec{k}}}{\partial(\hbar k_y)}k_y\frac{\partial f}{\partial\varepsilon_{\vec{k}}} = -\frac{ecE}{B}\int dk_x\int dk_y\, D_{\vec{k}}k_y\frac{\partial f}{\partial k_y} \\ &= \frac{ecE}{B}\int dk_x\int dk_y\, D_{\vec{k}}f = \frac{ecE}{B}\int[d\vec{k}]f = \frac{necE}{B} = -\frac{1}{\mathfrak{R}B}E, \qquad \mathfrak{R} = -\frac{1}{nec} \end{aligned} \tag{13}$$

when performing the y integration by parts, where $D_{\vec{k}} = 2/(2\pi)^2$ is a constant and the boundaries of the integral is the boundaries of the first Brillouin zone. (13) is the usual high-field result for the off-diagonal conductivity $\sigma_{yx} = -1/(\mathfrak{R}B)$.