Solutions to the problems in Chapter 27

27.1 Superconducting sphere

The free-energy density of a type-I superconductor is determined by the critical field H_c. Since $\vec{B} = \vec{0}$ throughout the volume of the type-I superconductor, when surface effects are neglected ($\lambda_L \approx 0$ compared to the dimensions of the sample)

$$\vec{B}_i = \vec{H}_i + 4\pi\vec{M} = \vec{0} \quad \Rightarrow \quad \vec{M} = -\frac{1}{4\pi}\vec{H}_i \tag{1}$$

Assuming the sample to be a thin needle along the direction of the applied magnetic field $\vec{H}_0$ (along z), the demagnetization field $\vec{H}_d = -\mathcal{N}_z\vec{M} \approx \vec{0}$. In this case, the magnetic field within the sample is $\vec{H}_i = \vec{H}_0$ and the magnetization is $\vec{M} = -\vec{H}_0/(4\pi)$. According to eq. (6) in the note "Magnetic energy and domains", the magnetic energy density is then,

$$\frac{1}{V}F_M = \mathcal{G}_M = -\int_{\text{sample}} \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\delta\vec{H}_0\right] = \frac{1}{4\pi}\int_0^{H_0} \vec{H}_0'\cdot d\vec{H}_0' = \frac{H_0^2}{8\pi} \tag{2}$$

[This energy density corresponds to $\mathcal{G}$ in Marder because $\vec{H}_0$ is the independent variable, see problem 27.2]. The total energy density is the sum of $\mathcal{G}_M$ and the field-independent condensation-energy density, $\mathcal{G}_S$. Since the two contributions just outbalance each other at the critical field $H_0 = H_c$, we have a measure for $\mathcal{G}_S$,

$$\mathcal{G} = \mathcal{G}_S + \mathcal{G}_M, \qquad \mathcal{G}_S = -\frac{H_c^2}{8\pi} \tag{3}$$

If the sample has instead the shape of a sphere, then (1) and (3) still apply, but

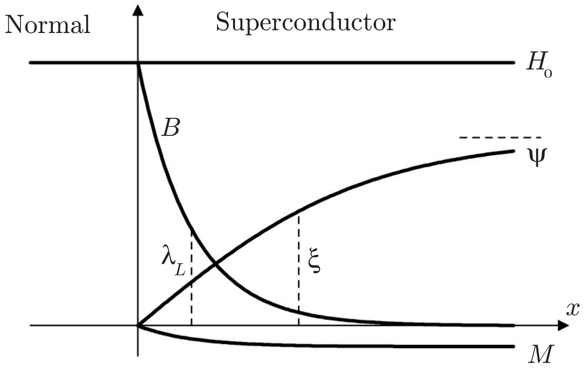
$$\vec{H}_i = \vec{H}_0 + \vec{H}_d = \vec{H}_0 - \mathcal{N}_z\vec{M} = \vec{H}_0 + D\vec{H}_i, \qquad D \equiv \frac{\mathcal{N}_z}{4\pi} = \frac{1}{3} \quad \text{(sphere)} \tag{4}$$

implying

$$\vec{H}_i = \frac{\vec{H}_0}{1-D}, \qquad \mathcal{G} = \mathcal{G}_S + \mathcal{G}_M = -\frac{H_c^2}{8\pi} + \frac{H_0^2}{8\pi(1-D)} \tag{5}$$

The internal field H_i is **larger** than the applied field and becomes equal the critical one, when the applied field is $H_0 = (1-D)H_c = \frac{2}{3}H_c$. This condition is not necessarily critical, since the global free energy of the superconducting sphere is still negative (as long as $H_0 < \sqrt{1-D}\,H_c$). A detail discussion of the *intermediate state* of the superconductor, when $(1-D)H_c < H_0 < H_c$, may be found in Landau and Lifshits, *Electrodynamics of continuous media* (Volume 8, 2nd Edition, page 189). In the intermediate range of the applied field, local thermodynamic instabilities are going to destroy superconductivity in parts of the sphere such that the effective D is reduced sufficiently to sustain superconductivity in the remaining parts. The assumption that the superconducting domain takes the form of a prolate ellipsoid with radius $a < R$ and constant height $2R$ (the largest volume for a certain value of D) is the most obvious one, but this configuration turns out to be thermodynamical unstable (the field in the normal part of the sphere is smaller than H_c near the end points of the ellipsoid). Nevertheless, domains of superconductivity are going to be present as long as $H_0 < H_c$.

27.2 Energy of normal-superconducting interfaces



(a) To start with we consider the limit of $\kappa \to 0$ or $\lambda_L \ll \xi$. Then $B = 0$ for $x > 0$ and according to (27.41) the wave function, which vanishes at $x = 0$ and saturates at Ψ_0 at large x, is

$$\Psi(x) = \Psi_0\, \psi(x) = \Psi_0 \tanh\left(\frac{x}{\sqrt{2}\xi}\right) \tag{1}$$

The free energy density, the Ginzburg–Landau energy is, (27.27),

$$\mathcal{F} = \Psi_0^2 \int_0^L \frac{dx}{L}\left[\alpha\, \psi^2 + \tfrac{1}{2}\beta\, \Psi_0^2 \psi^4 + \frac{\hbar^2}{2m^*}(\psi')^2\right] \tag{2}$$

Here $\Psi_0^2 = -\alpha/\beta$ is the equilibrium value at large x, and the change in energy per unit area, in comparison with the homogeneous case, is

$$L\Delta\mathcal{F} = \Psi_0^2 \int_0^\infty dx\left[\alpha\,(\psi^2 - 1) + \tfrac{1}{2}\beta\, \Psi_0^2(\psi^4 - 1) + \frac{\hbar^2}{2m^*}(\psi')^2\right] \tag{3}$$

Introducing $H_c^2 = 4\pi\alpha^2/\beta$, (27.35), and $\xi^2 = -\hbar^2/(2m^*\alpha)$, (27.31), then (3) may be written

$$L\Delta\mathcal{F} = \frac{H_c^2}{4\pi}\int_0^\infty dx\left[\tfrac{1}{2}\left(1 - \psi^2\right)^2 + \xi^2(\psi')^2\right] = \frac{H_c^2}{4\pi}\int_0^\infty dx\left(1 - \psi^2\right)^2 \tag{4}$$

using $\psi' = (1 - \psi^2)/(\sqrt{2}\xi)$, (27.40), in the last step. Applying this relation ones more, we get

$$L\Delta\mathcal{F} = \frac{H_c^2}{4\pi}\int_0^\infty dx\left(1 - \psi^2\right)\sqrt{2}\,\xi\psi' = \frac{H_c^2\sqrt{2}\,\xi}{4\pi}\int_0^1\left(1 - \psi^2\right)d\psi = \frac{H_c^2\sqrt{2}\,\xi}{4\pi}\,\frac{2}{3} \tag{5}$$

which energy per unit area is positive and in agreement with (27.52) [$\Delta\mathcal{G} = \Delta\mathcal{F}$ since $\vec{B} = \vec{0}$].

(b) Next we consider the opposite limit of $\xi \ll \lambda_L$, in which case $\Psi(x) \simeq \Psi_0$ and the gradient ψ' may be neglected. In this case the free energy density is

$$\mathcal{F} = \mathcal{F}_S + \mathcal{F}_M, \qquad \mathcal{F}_M = \int \frac{d\vec{r}}{V}\left(\frac{B^2}{8\pi} + \frac{4e^2}{2m^*c^2}A^2\Psi_0^2\right) = \int \frac{d\vec{r}}{8\pi V}\left(B^2 + \frac{A^2}{\lambda_L^2}\right) \tag{6}$$

where the first term is the zero-field contribution of the homogeneous superconductor, (2) when $\psi' = 0$, and $\mathcal{F}_M$ is the magnetic energy density in (27.27), when

$\lambda_L^2 = m^* c^2/(16\pi e^2 \Psi_0^2)$, (27.32), is inserted. The applied field is assumed to be parallel to the interface, i.e. along z, and $\vec{B} = \vec{H}_0$ in the normal metal [this configuration leads to the largest possible energy gain]. Assuming $\vec{A}$ to be along the y direction, $\vec{A} = (0, A(x), 0)$, then $\vec{B} = \nabla \times \vec{A} = (0, 0, A'(x))$ for $x > 0$. According to (27.9), $B_z = B(0)\, e^{-x/\lambda_L}$ for $x > 0$, where $B(0) = H_0$. Hence

$$\begin{aligned} &A'(x) = B(x) = H_0\, e^{-x/\lambda_L} \quad \Rightarrow \quad A(x) = -\lambda_L H_0\, e^{-x/\lambda_L} \; (\to 0 \text{ for } x \to \infty) \\ &\text{or} \qquad [A(x)/\lambda_L]^2 = B^2(x) = H_0^2\, e^{-2x/\lambda_L} \end{aligned} \tag{7}$$

introducing this in (6), we get (for the field energy within the superconductor)

$$L\,\mathcal{F}_M = \int_0^L dx\, \frac{2B^2(x)}{8\pi} = \int_0^\infty dx\, \frac{H_0^2}{4\pi}\, e^{-2x/\lambda_L} = \frac{H_0^2}{8\pi}\, \lambda_L \tag{8}$$

The magnetic free energy may also be calculated in a direct fashion

$$\mathcal{F}_M = \frac{1}{4\pi} \int \frac{d\vec{r}}{V} \left[\int \vec{H} \cdot \delta\vec{B} \right] = \frac{1}{4\pi} \int_0^\infty \frac{dx}{L} \left[\int_0^{H_0} H\, e^{-x/\lambda_L} dH \right] = \frac{H_0^2}{8\pi} \frac{\lambda_L}{L} \tag{9}$$

when using $\delta\vec{B} = e^{-x/\lambda_L} \delta\vec{H}$, and that the $\vec{H}$-field within the superconductor is the same as the applied field in the normal metal. In this free energy density $\vec{B}$ is the independent variable. The thermodynamical potential which is at a minimum in the equilibrium state, is the one where the applied magnetic field $\vec{H}$ is the independent variable. The relevant thermodynamic potential to be minimized is obtained by a Legendre transformation:

$$\begin{aligned} \mathcal{G}_M &= \mathcal{F}_M - \frac{1}{4\pi} \int \frac{d\vec{r}}{V} \vec{H} \cdot \vec{B} \quad \left(= -\frac{1}{4\pi} \int \frac{d\vec{r}}{V} \left[\int \vec{B} \cdot \delta\vec{H} \right] \right) \\ &= \mathcal{F}_M - \frac{1}{4\pi} \int_0^\infty \frac{dx}{L}\, H_0^2 e^{-\lambda_L/x} = \mathcal{F}_M - \frac{H_0^2}{4\pi} \frac{\lambda_L}{L} = -\frac{H_0^2}{8\pi} \frac{\lambda_L}{L} \end{aligned} \tag{10}$$

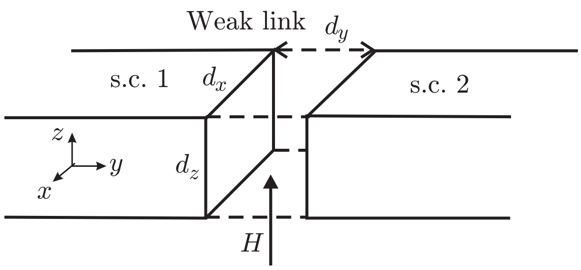
The magnetization in the superconductor is along the z axis and is determined from $\vec{B} = \vec{H} + 4\pi\vec{M}$ to be $M = M(x) = -(1 - e^{-x/\lambda_L}) H_0/(4\pi)$. The allowance of the magnetic $\vec{B}$-field to penetrate into the superconductor gives rise to the energy density gain determined by (10) in comparison with the case of $\vec{B} = \vec{0}$ (or $\lambda_L = 0$). Based on eq. (6) in the note "Magnetic energy and domains", this gain of magnetic energy density may be calculated in a direct fashion as

$$\begin{aligned} \frac{1}{V} \Delta F_M &= -\int \frac{d\vec{r}}{V} \left[\int \vec{M} \cdot \delta\vec{H} \right] + \int \frac{d\vec{r}}{V} \left[\int \vec{M} \cdot \delta\vec{H} \right]_{\vec{B}=\vec{0}} \\ &= \int_0^L \frac{dx}{L} \int_0^{H_0} dH \left(1 - e^{-x/\lambda_L}\right) \frac{H}{4\pi} - \int_0^L \frac{dx}{L} \int_0^{H_0} dH \frac{H}{4\pi} \\ &= \int_0^\infty \frac{dx}{L} \int_0^{H_0} dH \left(-e^{-x/\lambda_L}\right) \frac{H}{4\pi} = -\frac{H_0^2}{8\pi} \frac{\lambda_L}{L} = \mathcal{G}_M \end{aligned} \tag{11}$$

[The demagnetization field $\vec{H}_d = -\mathcal{N}\vec{M}$ is neglected in all the calculations, i.e. the sample is assumed to be a thin rod along the field, see problem 27.1]. In case the applied field is equal the critical field, $H_0 = H_c$, the interface is stable if $2\sqrt{2}\,\xi H_c^2/(12\pi) - \lambda_L H_c^2/(8\pi) < 0$ or if $\kappa > 4\sqrt{2}/3$.

27.3 Diffraction effects in Josephson junctions

The figure shows a weak link of normal metal (dashed lines) between two parts of a superconductor. The magnetic field $\vec{H}$ is applied in the z direction. $\vec{B} = \vec{0}$ within the superconductors, whereas $\vec{B} = \vec{H}$ in the normal metal. The magnetic flux through the weak link is $\Phi = \mathcal{A}B = \mathcal{A}H$, where the area is $\mathcal{A} = d_x d_y$.



(a) With the choice of the Landau gauge, the vector potential within the normal metal is

$$\vec{A} = (0, x\,H, 0), \quad \Rightarrow \quad \nabla \times \vec{A} = (0, 0, H) = \vec{B}, \qquad \nabla \cdot \vec{A} = 0 \tag{1}$$

(see also problem 25.4).

(b) The path integral, calculated at a constant x, between the two superconductors 1 and 2 is

$$\int_1^2 d\vec{s} \cdot \vec{A} = \int_0^{d_y} x\,H\,dy = H d_y\,x \tag{2}$$

The current density in the y direction between the two superconductors, (27.68a), is independent of z (when staying within the cross-section of the normal metal):

$$j(x) = j_0 \sin\left(\phi_2 - \phi_1 + \frac{2e}{\hbar c}\int_1^2 d\vec{s} \cdot \vec{A}\right) = j_0 \sin\left(\phi_2 - \phi_1 + \frac{4\pi}{\Phi_0} H d_y\,x\right) \tag{3}$$

The magnetic flux quantum is here defined, (25.51), $\Phi_0 = hc/e$, however, in much of the literature on superconductivity, the flux quantum is defined as the one applying for Cooper-pairs, i.e. $\Phi_0 = hc/e^*$. The total current is

$$\begin{aligned} J &= \int_0^{d_z} dz \int_0^{d_x} dx\, j(x) = d_z\, j_0 \left(\frac{\Phi_0}{4\pi H d_y}\right) \left[-\cos\left(\phi_2 - \phi_1 + \frac{4\pi H d_y}{\Phi_0}\,x\right)\right]_0^{d_x} \\ &= (d_z d_x) j_0 \frac{\Phi_0}{4\pi\Phi} \left\{\cos(\phi_2 - \phi_1) - \cos\left(\phi_2 - \phi_1 + \frac{4\pi\Phi}{\Phi_0}\right)\right\} \end{aligned} \tag{4}$$

when introducing the total magnetic flux $\Phi = d_x d_y H$. Defining $J_0 = d_z d_x j_0$ and using the general cosine relation: $\cos(u - v) - \cos(u + v) = 2\sin u \sin v$, we finally get

$$J = J_0 \frac{\Phi_0}{2\pi\Phi} \sin\left(\phi_2 - \phi_1 + \frac{2\pi\Phi}{\Phi_0}\right) \sin\left(\frac{2\pi\Phi}{\Phi_0}\right) \tag{5}$$

(c) At zero voltage the phases of the wave functions in the superconductors 1 and 2, ϕ_1 and ϕ_2, are constant in time. The result (5) clearly shows that the maximum current at a certain field and zero voltage is

$$J_{\max} = J_0 \frac{\Phi_0}{2\pi\Phi} \left|\sin\left(\frac{2\pi\Phi}{\Phi_0}\right)\right| \tag{6}$$

Comments concerning the Ginzburg–Landau free energy

The "Landau–Ginzburg" free energy presented by Marder in Eq. (27.27) is derived from the total magnetic contribution to the free energy density:

$$\mathcal{F}_M = \frac{1}{4\pi}\int \frac{d\vec{r}}{V}\left[\int \vec{H}\cdot\delta\vec{B}\right] = \frac{1}{8\pi}\int \frac{d\vec{r}}{V}B^2 - \int \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\delta\vec{B}\right] \tag{1}$$

by the use of $\vec{B} = \vec{H} + 4\pi\vec{M}$. Introducing the vector potential, $\vec{B} = \nabla\times\vec{A}$ and applying the vector identity $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot\nabla\times\vec{a} - \vec{a}\cdot\nabla\times\vec{b}$, the last integral may be written

$$\begin{aligned}&\int \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\delta\vec{B}\right] = \int \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\nabla\times\delta\vec{A}\right] = \\ &\int \frac{d\vec{r}}{V}\left[\int \delta\vec{A}\cdot\nabla\times\vec{M}\right] - \int \frac{d\vec{r}}{V}\nabla\cdot\left[\int(\vec{M}\times\delta\vec{A})\right] = \int \frac{d\vec{r}}{V}\left[\int \delta\vec{A}\cdot\nabla\times\vec{M}\right]\end{aligned} \tag{2}$$

Because of Gauss theorem the integral of the divergence may be written as an integral over a surface (far) outside the sample, where $\vec{M} = \vec{0}$, and this integral is zero. Introducing the internal current density $\vec{j}(\vec{r})$ of the sample, then $\nabla\times\vec{M} = \vec{j}/c$ and we finally get

$$\mathcal{F}_M = \int \frac{d\vec{r}}{V}\left[\frac{1}{8\pi}B^2 - \frac{1}{c}\int \vec{j}\cdot\delta\vec{A}\right] \tag{3}$$

in agreement with the magnetic part of $\mathcal{F}$ in (27.27), when introducing the superconducting current given by (27.29a). Formally, this expression is right, but one has to be careful with the integration of B^2 as the integral should really be performed over "all space" (the field outside the sample is modified when $\vec{M}$ is non-zero). Furthermore, the independent variable is $\vec{B}$ and not the applied field, and in order to change variable one has to perform a Legendre transformation:

$$\tilde{\mathcal{G}}_M = \mathcal{F}_M - \frac{1}{4\pi}\int \frac{d\vec{r}}{V}\vec{H}\cdot\vec{B} = -\frac{1}{4\pi}\int \frac{d\vec{r}}{V}\left[\int \vec{B}\cdot\delta\vec{H}\right] \tag{4}$$

Both the two complications are circumvented by applying instead the magnetic energy expression (6) derived in the note "Magnetic energy and domains":

$$\mathcal{G}_M = -\int \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\delta\vec{H}\right] \qquad \left(= \tilde{\mathcal{G}}_M + \frac{1}{8\pi}\int \frac{d\vec{r}}{V}H^2\right) \tag{5}$$

The integrand is only non-zero within the sample and $\vec{H}$ is the independent variable. [Here and in (4), we neglect the extra complication discussed in problem 27.1, that the internal field $\vec{H} = \vec{H}_i$ may differ from the applied one. Since the derivation of this energy does not involve a Legendre transformation, one may argue that this is the "Helmholtz" ($\mathcal{F}_M$) and not the "Gibbs" ($\mathcal{G}_M$) free energy]. Using (5) rather than (1) as the starting point then (3) is replaced by

$$\mathcal{G}_M = \int \frac{d\vec{r}}{V}\left[\int \vec{M}\cdot\delta(4\pi\vec{M}) - \int \vec{M}\cdot\delta\vec{B}\right] = \int \frac{d\vec{r}}{V}\left[2\pi M^2 - \frac{1}{c}\int \vec{j}\cdot\delta\vec{A}\right] \tag{6}$$

or $B^2/(8\pi)$ is being replaced by $2\pi M^2$ in the Ginzburg–Landau free energy expression (27.27).