2.2 The magnetic anisotropy

In this section, we shall discuss the thermal expectation-values of the Stevens operators of the single ions when their moments are non-zero, so that $|\langle \mathbf{J}_i \rangle| = \sigma J$. We shall then consider the contribution which the single-ion terms in the Hamiltonian make to the free energy, and thereby derive the relationship between the microscopic parameters and the macroscopic magnetic-anisotropy and magnetoelastic coefficients.

2.2.1 Temperature dependence of the Stevens operators

In a ferromagnet, the *Zener power-law* (1.5.15) for the expectation values of the Stevens operators is valid only at the lowest temperatures. Callen and Callen (1960, 1965) have derived $\langle O_l^m \rangle$ in exchange-dominated systems and obtained results which are useful over a much wider temperature range than the Zener expression. They begin with the density matrix for a single site in the MF approximation, including only the exchange and Zeeman energies,

$$\rho_{\mathrm{MF}}(x) = \frac{1}{Z}\exp(xJ_z/J) \qquad ; \quad x = \beta\{\mathcal{J}(\mathbf{0})J^2\sigma + g\mu_B JH\}, \quad (2.2.1)$$

where $\sigma = M/M_0$ is the relative magnetization, the direction of which is assumed to be parallel to the field. In this case the nth moment of J_z is determined as

$$\sigma_n = \langle (J_z/J)^n \rangle = \frac{1}{Z}\sum_{p=-J}^{J} \big(\frac{p}{J}\big)^n \exp(xp/J). \qquad (2.2.2)$$

This equation offers the possibility of relating the higher moments σ_n to the first moment, which is the relative magnetization $\sigma_1 = \sigma$, without referring explicitly to the MF value of x in eqn (2.2.1). According to the analysis of Callen and Shtrikman (1965), the functional dependence of σ_n on σ has a wider regime of validity than the MF approximation, because it only utilizes the exponential form of the density matrix, which is still valid when correlation effects are included in the random-phase approximation, where the excitations are collective spin-waves, as we shall discuss in Section 3.5. Furthermore, they found that the functions $\sigma_n = \sigma_n(\sigma)$; $n \geq 2$, derived from (2.2.2), only depend very weakly on the actual value of J, and for increasing values these functions rapidly converge towards the results obtained in the limit of infinite J (Callen and Callen 1965). In this limit, the sums in (2.2.2) are replaced by integrals, and the reduced diagonal matrix-elements of the Stevens operators are

$$(1/J^{(l)}) < J_z = p \,|\, O_l^m \,|\, J_z = p > \big|_{J\to\infty} = \delta_{m0} c_l P_l(u = p/J), \quad (2.2.3)$$

where the $J^{(l)}$ are defined by eqn (1.5.25), $P_l(u)$ are the Legendre polynomials, and c_l are constants. Multiplying the terms in the sum in (2.2.2) by $\Delta p = J\Delta u = 1$, and then taking the limit $J \to \infty$, we obtain

$$\frac{1}{c_l J^{(l)}}\langle O_l^0 \rangle = \int_{-1}^{1} P_l(u) e^{xu} du \Big/ \int_{-1}^{1} e^{xu} du = I_{l+\frac{1}{2}}(x)/I_{\frac{1}{2}}(x) = \hat{I}_{l+\frac{1}{2}}(x). \qquad (2.2.4)$$

$\hat{I}_{l+\frac{1}{2}}(x)$ is the usual shorthand notation for the ratio of $I_{l+\frac{1}{2}}(x)$ to $I_{\frac{1}{2}}(x)$, and the functions $I_{l+\frac{1}{2}}(x) = (-i)^{l+\frac{1}{2}} J_{l+\frac{1}{2}}(ix)$ are the modified spherical (or hyperbolic) Bessel functions. The relative magnetization

$$\sigma = \hat{I}_{\frac{3}{2}}(x) = \coth x - \frac{1}{x}$$

is the familiar Langevin function $\mathcal{L}(x)$ and, eliminating x in (2.2.4) by writing $x = \mathcal{L}^{-1}(\sigma)$, we finally arrive at

$$\langle O_l^m(\sigma) \rangle = \delta_{m0} c_l J^{(l)} \hat{I}_{l+\frac{1}{2}}[\sigma] \qquad \text{with} \quad \hat{I}_{l+\frac{1}{2}}[\sigma] = \hat{I}_{l+\frac{1}{2}}\big(\mathcal{L}^{-1}(\sigma)\big), \qquad (2.2.5)$$

for the thermal average of the Stevens operators as functions of $\sigma = \sigma(T, H)$, where $c_2 = 2$, $c_4 = 8$, and $c_6 = 16$. This result has turned out to be very useful for analysing the variation of the magnetic anisotropies and the magnetoelastic strains with temperature and magnetic field. In order to take full advantage of the theory, σ in eqn (2.2.5) is usually taken as the experimental value. If this is not available, it is a better approximation to use the correct MF value for it, rather than the Langevin-function, i.e. $\sigma = B_J(x)$ where $B_J(x)$ is the Brillouin function (1.2.31), determined directly from (2.2.2), because the actual value of J has some influence on the magnitude of σ. This is particularly true for the change of σ with magnetic field. In the limit of infinite J, $\partial\sigma/\partial(JH) \simeq (1-\sigma)g\mu_B/(J^2\mathcal{J}(\mathbf{0}))$ at low temperatures, which is just a factor of three smaller than the MF value for $J = 6$, which agrees reasonably well with experiments on Tb.

The functions $\hat{I}_{l+\frac{1}{2}}(x)$, for $l = 2, 3, \cdots$ are most easily calculated from the recurrence relation

$$\hat{I}_{l+\frac{3}{2}}(x) = \hat{I}_{l-\frac{1}{2}}(x) - \frac{2l+1}{x}\hat{I}_{l+\frac{1}{2}}(x). \tag{2.2.6}$$

At low temperatures, where $x \gg 1$ and $\sigma \simeq 1 - \frac{1}{x}$, it may easily be shown from (2.2.6) that $\hat{I}_{l+\frac{1}{2}}[\sigma] \simeq \sigma^{l(l+1)/2}$ (differences appear only in the third order of $m = 1-\sigma$). Hence the result (2.2.5) of the Callen–Callen theory agrees with the Zener power-law in the low-temperature limit. With increasing temperature, as x becomes comparable to 1, the exponential terms in the expansion of $\sigma \simeq 1 - \frac{1}{x} + 2\exp(-2x) + \cdots$, which have no counterpart in the classical Zener power-law, start to be important. In Chapter 5, we shall develop a detailed description of the excitations in the ferromagnet, the spin-waves. The thermal population of the spin-wave states is described by Bose statistics, assuming the availability of an infinite number of states of the single angular-momentum operators. The spin-wave theory reproduces the result of the Callen–Callen theory, in an expansion in powers of $m = 1 - \sigma$, but only if the exponential corrections above are negligible. The appearance of these terms at high temperatures signals the breakdown of the Bose approximation for the spin-wave excitations, which occurs because the actual number of states is not unlimited. As would be anticipated, this limitation in the number of states (or bandwidth if J is infinite) begins to be effective when the population of the uppermost level, which in the MF approximation is just proportional to $\exp(-2x)$, becomes significant. In the limit of a small relative magnetization, where $x \ll 1$, the Zener power-law and the spin-wave theory are both inadequate, whereas the Callen–Callen theory may still be applicable. In this limit, we may use the approximation

$$\hat{I}_{l+\frac{1}{2}}[\sigma] = \frac{3^l}{(2l+1)!!}\sigma^l \qquad ; \quad \sigma \ll 1. \tag{2.2.7}$$

One of the advantages of the Callen–Callen theory is that the results only depend on the one parameter σ, but not explicitly on the Hamiltonian. The relative magnetization may then be determined either by experiment, or by MF or more accurate theories, which result in a σ which depends on the actual Hamiltonian employed. The simplicity of this result may be impaired if the magnetic anisotropy of the system is substantial, so that the exchange interaction is no longer the dominant term in the density matrix. We shall be mostly concerned with the applicability of the theory at low temperatures, and the introduction of an axial-anisotropy term, such as $B_2^0 O_2^0(\mathbf{J}_i)$, is not inimical to the theory in this regime, provided that the magnetization is along the anisotropy c-axis, i.e. if B_2^0 is negative. Since only the lowest states are important at low temperatures and, in the MF approximation, these are still reasonably well accounted for by the density matrix in eqn (2.2.1), only the value of x is changed, with no direct consequences for the result. There are however noticeable effects if the anisotropy destroys the rotational symmetry about the magnetization axis. This is the case if B_2^0 is positive and forces the moments to lie in the basal plane, so that it requires a magnetic field to pull them out of it, whereas they may rotate much more freely within the plane, since B_6^6 is unimportant compared to the axial anisotropy. As we shall discuss in detail in Chapter 5, the ground state in this situation is no longer the fully-polarized state, the expectation value of J_z is slightly smaller than J at zero temperature, and the lower symmetry of the anisotropy field has direct consequences for the nature of the elementary spin-wave excitations, and thus for the form of the density matrix. The necessary modification of the Callen–Callen theory may be developed in two ways. One is to analyse the influence of the anisotropy on the low-temperature elementary excitations, and thereby derive the density matrix, as is done in Chapter 5. The other approach is numerical and involves the construction of a Hamiltonian which has the right transition temperature and the correct anisotropy fields, in the MF approximation. ρ_{MF} may then be calculated as a function of temperature, and results corresponding to (2.2.5), relating the expectation values of the various Stevens operators to the relative magnetization, may be obtained numerically. By the same argumentation as that used by Callen and Shtrikman (1965), these results may be expected to be insensitive to the actual model Hamiltonian used for describing the system. In the low temperature limit, the spin-wave theory supports this point of view, as its results are described in terms of only two parameters. One is the relative magnetization σ, as before, while the other, $\widetilde{b}$ or $\eta_\pm = (1 \pm \widetilde{b})(1 - \frac{1}{2}\widetilde{b}^2)$, measures the eccentricity of the anisotropic potential about the axis of magnetization.

In our discussion of the Callen–Callen theory, we have assumed that the quantization axis (z-axis), defining the Stevens operators, coincides with the direction of magnetization. We shall continue to use this convention, but must then take account of the difficulty that the crystal-field Hamiltonian in the hcp metals only has the simple form of eqn (1.4.6*b*) if the quantization axis is chosen to be along the c- or ζ-direction. In order to distinguish between the two systems, we shall denote the Stevens operators in the Hamiltonian defined with respect to the crystallographic axes, i.e. in the (ξ, η, ζ)-coordinate system, by $Q_l^m(\mathbf{J})$. The direction of magnetization, the z-axis, is specified by the polar angles (θ, ϕ) in the (ξ, η, ζ)-coordinate system, and we must introduce the following transformation of the angular momentum operators in $Q_l^m(\mathbf{J})$:

$$\begin{aligned} J_\xi &= J_z \sin\theta\cos\phi - J_x \cos\theta\cos\phi + J_y \sin\phi \\ J_\eta &= J_z \sin\theta\sin\phi - J_x \cos\theta\sin\phi - J_y \cos\phi \\ J_\zeta &= J_z \cos\theta + J_x \sin\theta, \end{aligned} \tag{2.2.8}$$

choosing the y-axis to lie in the basal-plane. By this transformation, Q_l^m is expressed as a linear combination of the Stevens operators $O_l^{m'}$, with the same l but various m'-values. For instance, we have

$$\begin{aligned} Q_2^0 &= 3J_\zeta^2 - J(J+1) \\ &= 3J_z^2\cos^2\theta + 3J_x^2\sin^2\theta + \tfrac{3}{2}(J_zJ_x + J_xJ_z)\sin 2\theta - J(J+1) \\ &= \tfrac{1}{2}O_2^0(3\cos^2\theta - 1) + \tfrac{3}{2}O_2^2\sin^2\theta + 3O_2^1\sin 2\theta. \end{aligned} \tag{2.2.9}$$

Carrying out the same transformation on Q_2^2 we find the following relations:

$$\begin{aligned} Q_2^0 &= \tfrac{1}{2}(-O_2^0 + 3O_2^2) \\ Q_2^2 &= \tfrac{1}{2}(O_2^0 + O_2^2) \qquad ; \quad \phi = p\pi, \end{aligned} \tag{2.2.10}$$

when the moment is in the basal-plane ($\theta = \pi/2$). The expectation value of Q_2^2 is relevant for determining the γ-strain $\epsilon_{\gamma 1}$, as shown in (1.5.29). According to the result (2.2.5) of Callen and Callen, $\langle O_2^2\rangle$ should vanish, but in a basal-plane ferromagnet this may not occur. The eccentricity parameter mentioned above is just defined as

$$\widetilde{b} = \langle O_2^2\rangle / \langle O_2^0\rangle, \tag{2.2.11}$$

which is zero, by definition, only if the anisotropy is invariant with respect to a rotation about the z-axis.

The numerical programme sketched above has been carried through for a model corresponding to Tb. The effective basal-plane anisotropy is about a factor of 10 smaller than the axial anisotropy, so that $\widetilde{b}$ is about -0.03 at zero temperature. The negative sign of $\widetilde{b}$ shows that the fluctuations of the moments within the plane are larger than those out of the plane, as measured respectively by $\langle J_y^2 \rangle$ and $\langle J_x^2 \rangle$, since $O_2^2 = J_x^2 - J_y^2$. In Fig. 2.2 the numerical results for $\langle O_2^0 \pm O_2^2 \rangle / J^{(2)}$ are compared with the predictions of the Callen–Callen theory, and of the linear spin-wave theory developed in Chapter 5, in which the MF values (5.3.23) of m_o and b_o are used, instead of (5.3.18). The Callen–Callen theory predicts that both thermal averages vary like $\hat{I}_{5/2}(\sigma)$, which is not consistent with a $\widetilde{b}$ different from zero. Furthermore, the effective power-laws in the zero-temperature limit are changed from σ^3 to $\langle O_2^0 - O_2^2 \rangle \propto \sigma^{2.65}$, and $\langle O_2^0 + O_2^2 \rangle \propto \sigma^{3.3}$. The predictions of the spin-wave theory are consistent with the numerical results at low temperatures, both with respect to the absolute magnitude of the expectation values and to the effective power-laws, and it appears to give a reasonably correct description of

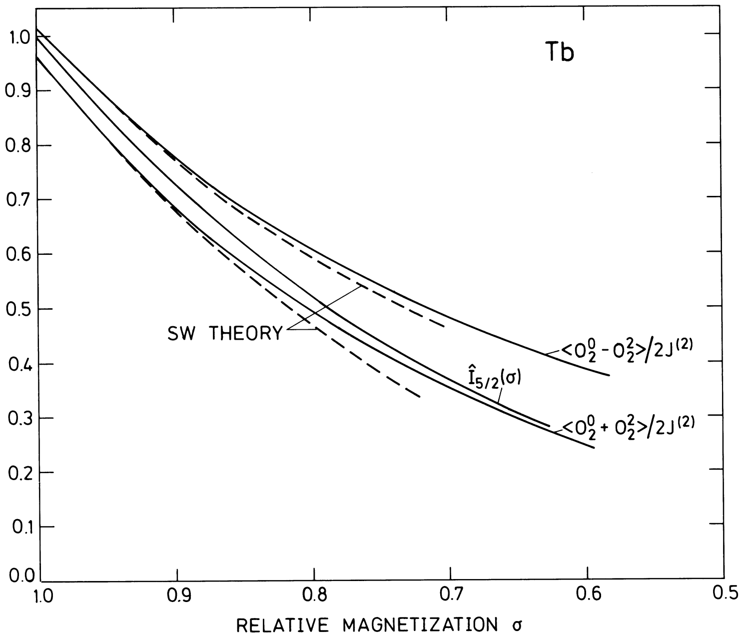

Fig. 2.2. Calculations of the dependence of the expectation values of the Stevens operators $\langle O_2^0 \pm O_2^2 \rangle$ on the relative magnetization in Tb. The numerical calculations described in the text differ from the Callen–Callen result $\hat{I}_{5/2}(\sigma)$, but agree at low temperatures with the predictions of spin-wave theory.

the system as long as σ is greater than about 0.8. The same picture holds true for other combinations of Stevens operators, but the discrepancies between the different theories are accentuated as the rank increases. Figure 2.3 shows the example of $\langle Q_6^6 \rangle$. The absolute magnitude of this quantity is reduced by nearly 40% in the zero-temperature limit, as compared with the Callen–Callen theory, and the slope of the numerical calculation, in the semi-logarithmic plot, changes with σ, leading to an effective power-law depending on the interval over which it is measured. In the zero-temperature limit, $\langle Q_6^6 \rangle$ is proportional to approximately σ^{26}, instead of the Callen–Callen result σ^{21}.

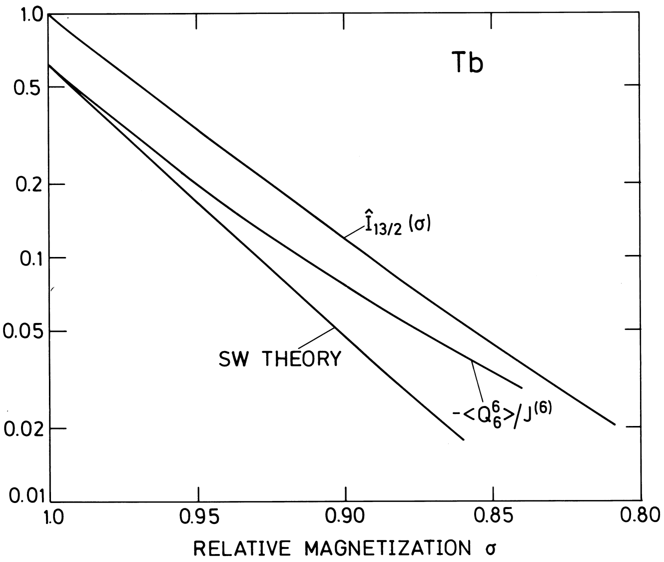

Fig. 2.3. The dependence on the relative magnetization of the expectation value of the Stevens operator $\langle Q_6^6 \rangle$, which determines the hexagonal magnetic anisotropy, in Tb. The numerical calculations and the spin-wave theory both predict a large reduction in this quantity at low temperatures, compared with the Callen–Callen theory.

The numerical results are expected to be sensitive to the magnitude of the anisotropy, rather than to the actual parameters which determine the anisotropy, and the spin-wave theory indicates that this expectation is fulfilled, at least at low temperatures. However, in order to obtain the right variation of the anisotropy fields with temperature, i.e. of $\widetilde{b}$ compared to σ, it is necessary to select appropriate linear combinations of Stevens operators of various ranks for the modelling of the different anisotropy terms. At high temperatures, for instance, $\widetilde{b}$ is determined by the low-rank terms alone, i.e. by B_2^0 if anisotropic dipole–dipole coupling is neglected. Using $\beta = (k_B T)^{-1}$ as an expansion parameter, and assuming the magnetization to lie in the basal plane, we find, to leading order in the crystal-field parameters introduced in eqn (1.4.6*b*),

$$\langle Q_2^2 \rangle = \tfrac{3}{5} J^{(2)} \sigma^2 \frac{J + \frac{3}{2}}{J + 1} \simeq J^{(2)} \hat{I}_{5/2}[\sigma], \qquad (2.2.12a)$$

using (2.2.7) and neglecting the small $1/J$ corrections, whereas

$$\langle Q_2^0 \rangle = -\langle Q_2^2 \rangle - \tfrac{4}{5} J^{(2)} (J+1)(J+\tfrac{3}{2}) \beta B_2^0, \qquad (2.2.12b)$$

which depends on the anisotropy, but only on the term of lowest rank. Considering the *field* dependence of the two expectation values, as determined by their dependence on σ, we observe that the Callen–Callen theory leads to the right result in this high-temperature limit. The two relations above explain the behaviour of $\langle O_2^0 \pm O_2^2 \rangle$ in Fig. 2.2, when σ becomes small, as $\langle O_2^0 + O_2^2 \rangle / 2J^{(2)}$ should approach $\hat{I}_{5/2}(\sigma)$ at small values of σ, and go to zero at the transition temperature. $\langle O_2^0 - O_2^2 \rangle / 2J^{(2)}$, on the other hand, should still be non-zero (about 0.23 as determined by $T_N \simeq 229\,\mathrm{K}$ and the value $B_2^0 = 0.18\,\mathrm{meV}$ used in the model) when T_N is approached from below and σ vanishes.

2.2.2 Anisotropic contributions to the free energy

The anisotropy of a magnetic system is determined by those contributions to the free energy which depend on the polar angles (θ, ϕ), which specify locally the direction of the moments. Restricting ourselves to the case of a ferromagnet in a uniform field, we may expand the free energy in terms of functions proportional to the spherical harmonics, as in eqn (1.5.22). To relate this expansion to the Hamiltonian (2.1.1), we may use (2.1.5), which states that any change in the free energy due to a change of the angles is given by $\delta\tilde{F} = \big\langle \delta\mathcal{H} \big\rangle$. The field is the independent variable in $\tilde{F}$ but, as in (2.1.22), we wish the magnetization to be the independent variable. To obtain this free energy $F(\theta, \phi)$, we subtract the Zeeman energy, so that $F(\theta, \phi) = \tilde{F} - \langle \mathcal{H}_\mathrm{Z} \rangle$, where the field needed for establishing the specified angles is determined from the equilibrium condition $\delta\tilde{F} = 0$. In the ferromagnet, the moments all point in the same direction, and any contributions from the isotropic-exchange coupling cancel out in $\delta\mathcal{H}$. The free-energy function $F(\theta, \phi)$ is thus determined by

$$\delta F(\theta, \phi) = \big\langle \delta(\mathcal{H}_\mathrm{cf} + \mathcal{H}_\mathrm{Z}) \big\rangle - \delta \big\langle \mathcal{H}_\mathrm{Z} \big\rangle. \qquad (2.2.13)$$

Introducing the angle variables in the Hamiltonian by the transformation (2.2.8), we find that the operators of rank l become angle-dependent linear combinations of the l-rank Stevens operators, which have their polar axis along the z-axis defined by the direction of the moments. The variational expression for the free energy then involves the calculation of the expectation values of these Stevens operators. To leading order in the crystal-field parameters, we may neglect the influence of the anisotropy terms on the thermal averages in (2.2.13). This is the approximation used by Callen and Callen, and we may utilize their result, eqn (2.2.5). This has the consequence that, in the various linear combinations of Stevens operators, only those terms in which $m = 0$ contribute to the free energy, to leading order in the anisotropy parameters. From the expansion (2.2.9) of Q_2^0, we find the following result:

$$\big\langle \delta\big(Q_2^0\big)\big\rangle \simeq \langle O_2^0\rangle \delta\big\{ \tfrac{1}{2}(3\cos^2\theta - 1)\big\}$$

and, repeating this calculation for the other operators, we have in general

$$\big\langle \delta\big(Q_l^0\big)\big\rangle \simeq \langle O_l^0\rangle \delta P_l(\cos\theta) \quad ; \quad \big\langle \delta\big(Q_6^6\big)\big\rangle \simeq \tfrac{1}{16}\langle O_6^0\rangle \delta\big\{ \sin^6\theta\cos 6\phi\big\}. \tag{2.2.14}$$

Because $\langle J_x\rangle = \langle J_y\rangle = 0$, the Zeeman terms in (2.2.13) cancel within this approximation, and an integration of $\delta F(\theta,\phi)$ leads to

$$F(\theta,\phi)/N \simeq f_0 + \sum_l B_l^0\langle O_l^0\rangle P_l(\cos\theta) + \tfrac{1}{16}B_6^6\langle O_6^0\rangle \sin^6\theta\cos 6\phi. \tag{2.2.15}$$

Comparing this result with the free energy expression (1.5.22), and introducing the anisotropy parameters $\kappa_l^m(T)$, we obtain to a first approximation

$$\kappa_l^0(T) = c_l B_l^0 J^{(l)}\hat{I}_{l+\frac{1}{2}}[\sigma] \qquad ; \quad \kappa_6^6(T) = B_6^6 J^{(6)}\hat{I}_{13/2}[\sigma], \tag{2.2.16}$$

with $\sigma = \sigma(T)$, which leads to eqn (1.5.24) at zero temperature ($\sigma = 1$).

The equilibrium values of the angles in zero field are determined by $\partial F(\theta,\phi)/\partial\theta = \partial F(\theta,\phi)/\partial\phi = 0$. In the above result for the free energy, the ϕ-dependence is determined exclusively by B_6^6, the sign of which then determines whether the a- or b-directions are the magnetically easy or hard axes in the basal-plane ($\phi_0 = p\pi/3$ or $\pi/2 + p\pi/3$). Because B_6^6 is a sixth-rank coupling parameter, the importance of this anisotropy decreases rapidly with the magnetization; $\hat{I}_{13/2}[\sigma] \propto \sigma^{21}$ at low temperatures, or σ^6 when σ is small. The axial anisotropy derives from all four parameters, and the equilibrium value θ_0 is determined by minimizing

$$\begin{aligned} f(u = \cos\theta) &= F(\theta,\phi_0)/N - f_0 \\ &= \tfrac{1}{2}\kappa_2^0(3u^2 - 1) + \tfrac{1}{8}\kappa_4^0(35u^4 - 30u^2 + 3) \\ &\quad + \tfrac{1}{16}\kappa_6^0(231u^6 - 315u^4 + 105u^2 - 5) - |\kappa_6^6|(1 - u^2)^3. \end{aligned} \tag{2.2.17}$$

Equation (2.2.16) shows that the various anisotropy parameters depend differently on temperature. At high temperatures, κ_2^0 dominates and its sign determines whether the moments are parallel or perpendicular to the c-axis. As the temperature is decreased, the importance of the higher-rank terms grows, putting increasing weight on the terms of fourth and sixth power in $\cos\theta$. The equilibrium value $\theta_0(T)$ of θ may therefore change as a function of temperature, as occurs in Ho and Er, and also in Gd where, however, the theory of this section is not immediately applicable.

The coefficients in the expansion for the free energy may be obtained from experimental studies of the magnetization as a function of the magnitude and direction of an applied magnetic field. The axial part of the anisotropy is predominantly determined by the three κ_l^0-parameters, and it is not usually easy to separate their contributions. At low temperatures, where the higher-rank terms become relatively important, the axial anisotropy in the heavy rare earths is frequently so strong that it is only possible to change θ by a few degrees from its equilibrium value. Under these circumstances, it is only possible to measure the components of the susceptibility, allowing a determination of the second derivatives of $F(\theta,\phi)$ in the equilibrium state $(\theta,\phi) = (\theta_0,\phi_0)$. The x-axis lies in the symmetry z–ζ plane and the transverse part of the susceptibility tensor is diagonal with respect to the (x,y)-axes. With a field h_x applied in the x-direction, the moments rotate through an angle $\delta\theta = \theta - \theta_0$, giving a component $\langle J_x\rangle = -\langle J_z\rangle\delta\theta = \chi_{xx}h_x$. Introducing the notation $F_{\theta\theta} \equiv \partial^2 F(\theta,\phi)/\partial\theta^2$ at $(\theta,\phi) = (\theta_0,\phi_0)$, and similarly for the other second derivatives, we may write

$$F = F(\theta_0,\phi_0) + \tfrac{1}{2}F_{\theta\theta}(\delta\theta)^2 + \tfrac{1}{2}F_{\phi\phi}(\delta\phi)^2 + N\langle J_z\rangle\delta\theta h_x,$$

in the limit where the field goes to zero. The term $F_{\theta\phi} = 0$, because $\sin 6\phi_0 = 0$. At equilibrium, $\delta\theta = -N\langle J_z\rangle h_x/F_{\theta\theta}$, which determines the susceptibility. When the field is applied in the y-direction, i.e. along the direction $(-\sin\phi_0, \cos\phi_0, 0)$, the Zeeman contribution to F is

$$N\langle J_z\rangle h_y \sin\theta\sin(\phi-\phi_0) = N\langle J_z\rangle h_y \sin\theta_0\delta\phi,$$

with $\langle J_y\rangle = -\langle J_z\rangle\sin\theta_0\delta\phi = \chi_{yy}h_y$. Minimizing the free energy in the presence of a field along the y-axis, we may derive the other susceptibility component, obtaining

$$\chi_{xx} = N\langle J_z\rangle^2/F_{\theta\theta} \qquad ; \qquad \chi_{yy} = N\langle J_z\rangle^2\sin^2\theta_0/F_{\phi\phi}. \tag{2.2.18}$$

In calculating χ_{yy}, we have assumed that $\theta_0 \neq 0$; if $\theta_0 = 0$ then $\chi_{yy} = \chi_{xx}$. Equation (2.2.18) is also valid in the presence of an external field, provided that the effects due to the Zeeman contribution,

$F_Z = -Ng\mu_B \mathbf{H}\cdot\langle\mathbf{J}\rangle$, are included explicitly in $F(\theta_0, \phi_0)$ and its derivatives. Introducing the expression (2.2.17) for the free energy, in the two cases where the moments are either parallel or perpendicular to the c-axis, we find

$$1/\chi_{xx} = 1/\chi_{yy} = -(3\kappa_2^0 + 10\kappa_4^0 + 21\kappa_6^0)/(\sigma J)^2 \qquad ; \quad \theta_0 = 0, \quad (2.2.19a)$$

or

$$\begin{aligned} 1/\chi_{xx} &= (3\kappa_2^0 - \tfrac{15}{2}\kappa_4^0 + \tfrac{105}{8}\kappa_6^0 + 6|\kappa_6^6|)/(\sigma J)^2 \\ 1/\chi_{yy} &= 36|\kappa_6^6|/(\sigma J)^2 \end{aligned} \qquad ; \quad \theta_0 = \frac{\pi}{2}, \qquad (2.2.19b)$$

which must be positive if the structure is to be stable. In order to determine the higher derivatives of the free energy, a transverse field greater than that corresponding to the linear regime described by the (zero-field) susceptibility must be applied. The application of a large magnetic field perpendicular to the magnetization axis, in a strongly anisotropic system, creates a large mechanical torque, which may cause practical problems with maintaining the orientation of the crystal. If the experimental facilities do not allow the determination of the higher derivatives, the different temperature dependences of the various anisotropy parameters may yield a rough separation of their contributions to the total axial anisotropy. However the Callen–Callen theory is an approximation, the corrections to which are important if the anisotropy is large, and there are other contributions to the free energy than those which we have considered above.

The results derived above are only valid if the anisotropy energies are small compared to the exchange energy. In order to demonstrate the kind of modifications which may appear in higher order, we shall consider the simplest possible case, where only B_2^0 is non-zero, and we shall only calculate the free energy at zero temperature in the MF approximation, i.e. the ground-state energy of a single site subjected to the exchange field $h_{\rm ex} = \langle J_z\rangle\mathcal{J}(\mathbf{0})$, with $\langle J_z\rangle = \sigma J$. In this case, the MF Hamiltonian (2.1.16) is

$$\begin{aligned} \mathcal{H} &= -(J_z - \tfrac{1}{2}\sigma J)\sigma J\mathcal{J}(\mathbf{0}) - h(J_z\cos\theta + J_x\sin\theta) \\ &\quad + B_2^0\big[3J_z^2\cos^2\theta + 3J_x^2\sin^2\theta + \tfrac{3}{2}(J_zJ_x + J_xJ_z)\sin 2\theta - J(J+1)\big], \end{aligned} \qquad (2.2.20)$$

in an applied field h along the ζ-axis. With the J_z-eigenstates as the basis, the leading-order ground-state energy is

$$E_0^0 = <J\,|\,\mathcal{H}\,|\,J> = -(1-\tfrac{1}{2}\sigma)\sigma J^2\mathcal{J}(\mathbf{0}) - hJ\cos\theta + B_2^0 J^{(2)}(3\cos^2\theta - 1).$$

The off-diagonal matrix elements involving the ground state are

$$\begin{aligned} <J-1\,|\,\mathcal{H}\,|\,J> &= \big\{6(J-\tfrac{1}{2})B_2^0\cos\theta - h\big\}(J/2)^{1/2}\sin\theta \\ <J-2\,|\,\mathcal{H}\,|\,J> &= \tfrac{3}{2}\{2J^{(2)}\}^{1/2}B_2^0\sin^2\theta. \end{aligned}$$

We shall only be concerned with terms up to second order in B_2^0 and h, so that we may use second-order perturbation theory, and it is sufficiently accurate to approximate the energy differences between the ground state and the first and second excited-states by respectively $\Delta_1 = J\mathcal{J}(\mathbf{0})$ and $\Delta_2 = 2J\mathcal{J}(\mathbf{0})$. Because of the mixing of the states, $\sigma = \langle J_z\rangle/J = 1-m$ becomes slightly smaller than 1, but this only affects the exchange contribution quadratic in m, as $(1-\frac{1}{2}\sigma)\sigma = \frac{1}{2}(1-m^2)$. To second order, the ground-state energy is found to be

$$\begin{aligned} E_0(h) = &-\tfrac{1}{2}J^2\mathcal{J}(\mathbf{0}) - hJ\cos\theta + B_2^0J^{(2)}(3\cos^2\theta-1) \\ &-\tfrac{1}{2}\big\{6(J-\tfrac{1}{2})B_2^0\cos\theta - h\big\}^2\sin^2\theta/\mathcal{J}(\mathbf{0}) - \tfrac{9}{4}(J-\tfrac{1}{2})(B_2^0)^2\sin^4\theta/\mathcal{J}(\mathbf{0}). \end{aligned} \tag{2.2.21}$$

The minimum condition $\partial E_0/\partial\theta = 0$ leads to

$$h = h_0 = 6(J-\tfrac{1}{2})B_2^0\big[1 + 3B_2^0\sin^2\theta/\{2J\mathcal{J}(\mathbf{0})\}\big]\cos\theta \qquad \text{or} \qquad \sin\theta = 0,$$

to second order in B_2^0. The free energy $F(\theta,\phi)$ at zero temperature is then, in both cases,

$$\begin{aligned} F(\theta,\phi)/N &= E_0(h_0) + h_0J\sigma\cos\theta \\ &= -\tfrac{1}{2}J^2\mathcal{J}(\mathbf{0}) + \tfrac{1}{2}\tilde{\kappa}_2^0(3\cos^2\theta-1) + \tfrac{3}{4}b\tilde{\kappa}_2^0\sin^4\theta, \end{aligned}$$

with

$$\tilde{\kappa}_2^0 = 2B_2^0J^{(2)} \qquad ; \quad b = -3B_2^0/\{2J\mathcal{J}(\mathbf{0})\}, \tag{2.2.22a}$$

and the relative magnetization is $\sigma = 1-(J-\frac{1}{2})b^2\sin^4\theta$. The b-parameter introduced here is the leading order contribution to $\widetilde{b}$, defined in (2.2.11), when $\theta = \pi/2$. One important feature illustrated by this calculation is that the O_2^m-term in Q_2^0, with m odd, is cancelled by the Zeeman contribution, to second order in B_2^0. This is a consequence of the freedom to replace the equilibrium condition $\partial F/\partial\theta = 0$ by the requirement that $\langle J_x\rangle$ (and $\langle J_y\rangle$) should vanish, by definition, with the implication that the matrix-element $<J-1\,|\,\mathcal{H}\,|\,J>$ must be zero within the present approximation. Bowden (1977) did not take the Zeeman effect into account, and therefore obtained an erroneously strong renormalization of the anisotropy. The second derivatives of $F(\theta,\phi)$ are $F_{\phi\phi} = 0$, and

$$F_{\theta\theta}/N = -3\tilde{\kappa}_2^0(1-b\sin^2\theta)\cos 2\theta + \tfrac{3}{2}\tilde{\kappa}_2^0 b\sin^2 2\theta. \tag{2.2.22b}$$

There is no change in the axial susceptibility in the axial ferromagnet, for which $\theta = 0$, but the higher derivatives are affected by the modifications $\kappa_2^0(0) = \tilde{\kappa}_2^0(1 - \frac{4}{7}b)$ and $\kappa_4^0(0) = \frac{6}{35}b\tilde{\kappa}_2^0$. The correction to the Callen–Callen theory is proportional to b, which is of the order $1/J$ times the ratio between the anisotropy and the exchange energy ($\propto B_2^0 J^{(2)}/J^2\mathcal{J}(\mathbf{0})$), and hence becomes smaller for larger values of J. This calculation may be extended to higher order and to non-zero temperatures, but the complications are much reduced by the application of the *Holstein–Primakoff transformation* which utilizes directly the factor $1/J$ in the expansion, as we shall see in the discussion of the spin-wave theory in Chapter 5.

In the ferromagnetic phase, the ordered moments may distort the lattice, due to the magnetoelastic couplings, and this gives rise to additional contributions to $F(\theta, \phi)$. We shall first consider the effects of the γ-strains by including the magnetoelastic Hamiltonian, incorporating (1.4.8) and (1.4.11),

$$\begin{aligned}\mathcal{H}_\gamma = \sum_i \big[\tfrac{1}{2}c_\gamma(\epsilon_{\gamma1}^2 + \epsilon_{\gamma2}^2) - B_{\gamma2}\big\{Q_2^2(\mathbf{J}_i)\epsilon_{\gamma1} + Q_2^{-2}(\mathbf{J}_i)\epsilon_{\gamma2}\big\} \\ - B_{\gamma4}\big\{Q_4^4(\mathbf{J}_i)\epsilon_{\gamma1} - Q_4^{-4}(\mathbf{J}_i)\epsilon_{\gamma2}\big\}\big],\end{aligned} \tag{2.2.23}$$

retaining only the lowest-rank couplings ($l = 2$ and 4 of respectively the $\gamma2$ and $\gamma4$ terms). The equilibrium condition

$$\partial F/\partial\epsilon_{\gamma1} = \big\langle \partial\mathcal{H}_\gamma/\partial\epsilon_{\gamma1}\big\rangle = 0, \tag{2.2.24}$$

and similarly for $\epsilon_{\gamma2}$, leads to the equilibrium strains

$$\begin{aligned}\epsilon_{\gamma1} &= \big(B_{\gamma2}\langle Q_2^2\rangle + B_{\gamma4}\langle Q_4^4\rangle\big)/c_\gamma \\ \epsilon_{\gamma2} &= \big(B_{\gamma2}\langle Q_2^{-2}\rangle - B_{\gamma4}\langle Q_4^{-4}\rangle\big)/c_\gamma.\end{aligned} \tag{2.2.25}$$

The conventional magnetostriction parameters C and A are introduced via the equations

$$\begin{aligned}\epsilon_{\gamma1} &= C\sin^2\theta\cos2\phi - \tfrac{1}{2}A\sin^4\theta\cos4\phi \\ \epsilon_{\gamma2} &= C\sin^2\theta\sin2\phi + \tfrac{1}{2}A\sin^4\theta\sin4\phi\end{aligned} \tag{2.2.26a}$$

(Mason 1954). Expressing Q_l^m in terms of O_l^m, and retaining only the terms with $m = 0$, we may derive these parameters from (2.2.25), obtaining

$$\begin{aligned}C &= \frac{1}{c_\gamma}B_{\gamma2}J^{(2)}\hat{I}_{5/2}[\sigma] \\ A &= -\frac{2}{c_\gamma}B_{\gamma4}J^{(4)}\hat{I}_{9/2}[\sigma].\end{aligned} \tag{2.2.26b}$$

Within this approximation, the γ-strain contribution $F_\gamma(\theta,\phi)$ to the free energy is

$$\begin{aligned} F_\gamma(\theta,\phi) &= \langle \mathcal{H}_\gamma \rangle = -\tfrac{1}{2}c_\gamma\big(\epsilon_{\gamma 1}^2+\epsilon_{\gamma 2}^2\big)N \\ &= -\tfrac{1}{2}c_\gamma\big(C^2\sin^4\theta+\tfrac{1}{4}A^2\sin^8\theta - CA\sin^6\theta\cos 6\phi\big)N, \end{aligned} \tag{2.2.27}$$

showing that these strains affect the axial-anisotropy parameters $\kappa_l^0(T)$, introducing effects of higher rank than $l=6$, and that the six-fold anisotropy in the basal plane is now

$$\kappa_6^6(T) = B_6^6 J^{(6)} \hat{I}_{15/2}[\sigma] + \tfrac{1}{2}c_\gamma CA. \tag{2.2.28}$$

When both C and A are non-zero, the maximum area-conserving elongation of the hexagonal planes varies between $|C+\frac{1}{2}A|$ and $|C-\frac{1}{2}A|$, which results in a ϕ-dependent magnetoelastic energy, and thus a contribution to κ_6^6. The γ-strain hexagonal anisotropy decreases more slowly (like σ^{13} at low temperatures) than the B_6^6 term, as σ decreases, and therefore dominates at sufficiently high temperatures.

The ε-strains may be included in a similar way. Retaining only the lowest-rank coupling $B_{\varepsilon 1} \equiv B_{\varepsilon 1}^{(l=2)}$ in eqn (1.4.12), we have

$$\mathcal{H}_\varepsilon = \sum_i \big[\tfrac{1}{2}c_\varepsilon(\epsilon_{\varepsilon 1}^2+\epsilon_{\varepsilon 2}^2) - B_{\varepsilon 1}\big\{Q_2^1(\mathbf{J}_i)\epsilon_{\varepsilon 1} + Q_2^{-1}(\mathbf{J}_i)\epsilon_{\varepsilon 2}\big\}\big]. \tag{2.2.29}$$

Introducing the magnetostriction parameter H_ε of Mason (1954) (the index ε should prevent any confusion with the magnetic field) by

$$\epsilon_{\varepsilon 1} = \tfrac{1}{4}H_\varepsilon \sin 2\theta\cos\phi \quad ; \quad \epsilon_{\varepsilon 2} = \tfrac{1}{4}H_\varepsilon \sin 2\theta \sin\phi, \tag{2.2.30a}$$

we obtain within the Callen–Callen theory

$$H_\varepsilon = \frac{2}{c_\varepsilon} B_{\varepsilon 1} J^{(2)} \hat{I}_{5/2}[\sigma], \tag{2.2.30b}$$

and the ε-strain contribution to the free energy

$$F_\varepsilon(\theta,\phi) = -\tfrac{1}{32}Nc_\varepsilon H_\varepsilon^2 \sin^2 2\theta. \tag{2.2.31}$$

The α-strains (1.4.10) do not influence the symmetry of the system, but they do make a contribution, essentially proportional to $\langle Q_2^0\rangle$, to the anisotropy, the effects of which may be derived in the same way as those of the γ- and ε-strains. The magnetoelastic contributions to the free energy can be estimated experimentally if the elastic constants are known, by a determination of the strains as a function of the magnetization. The knowledge of the equilibrium strains may also be used for a reasonable estimate of the magnetoelastic modifications of the second derivatives, provided that the additional assumption is made that the couplings of lowest rank are dominant. For example, the higher-rank γ-strains in the basal-plane magnet make contributions to the axial anisotropy which cannot be written in terms of C and A in eqn (2.2.27). A more direct estimate of the contributions to the second derivatives requires an experimental determination of how the strains behave when the direction of the magnetization is changed. In basal-plane ferromagnets, such as Tb and Dy, it may be possible to observe the ϕ-dependence of the strains (Rhyne and Legvold 1965a), whereas if the axial anisotropy is large, it may be very difficult to determine the variation of the strains with θ. In the case of the α-strains, the argument that the $(l = 2)$ couplings are dominant is not sufficient for a determination of their effect on the axial anisotropy. The reason is that the *two-ion magnetoelastic couplings* of lowest rank, i.e. the dipolar interactions

$$\begin{aligned}\Delta\mathcal{H}^{\alpha}_{\mathrm{me}} = -\sum_{ij}\Big[&\big\{D_{10}(ij)\epsilon_{\alpha 1} + D_{20}(ij)\epsilon_{\alpha 2}\big\}\,\mathbf{J}_i\cdot\mathbf{J}_j \\ &+ \big\{D_{13}(ij)\epsilon_{\alpha 1} + D_{23}(ij)\epsilon_{\alpha 2}\big\}J_{i\zeta}J_{j\zeta}\Big],\end{aligned} \tag{2.2.32}$$

may be important. This is the case in Tb and Dy, as shown by the analysis of the stress-dependence of the Néel temperatures (Bartholin *et al.* 1971). These interactions affect the α-strains, but they contribute differently to the axial anisotropy from the $(l = 2)$-terms in the single-ion magnetoelastic Hamiltonian (1.4.10).

The simplifications introduced in the above discussion of the ferromagnet may also be utilized in non-uniform systems, because the MF approximation allows the individual ions to be treated separately. However, the isotropic two-ion contributions no longer cancel in $\delta F(\theta,\phi)$ in (2.2.13), since the direction of the exchange field depends on the site considered. We consider as an example the helically ordered phase. If we neglect the bunching effect due to the hexagonal anisotropy, the axial anisotropy is independent of the site considered. Treating the ions as isolated, but subject to a constant exchange-field, we may calculate $F^{o}_{\theta\theta}$, corresponding to $1/\chi^{o}_{xx}$, and then use (2.1.19) to account for the induced exchange-field due to an applied field in the x- or c-direction, modulated with a wave-vector $\mathbf{q}$ along the c-axis. If the two-ion coupling between the moments is allowed to be anisotropic, the leading order result is

$$1/\chi_{xx}(\mathbf{q}) = \mathcal{J}_{\perp}(\mathbf{Q}) - \mathcal{J}_{\|}(\mathbf{q}) + \big(3\kappa^0_2 - \tfrac{15}{2}\kappa^0_4 + \tfrac{105}{8}\kappa^0_6\big)/(\sigma J)^2. \tag{2.2.33}$$

This is the anisotropy parameter which determines the plane in which the moments spiral, and it vanishes at the temperature T'_N at which the c-axis moments begin to order. Just below T_N', the c-component is modulated with the wave-vector $\mathbf{Q}'$ at which $\mathcal{J}_{\|}(\mathbf{q})$ has its maximum, and only if $\mathbf{Q}' = \mathbf{Q}$ is the structure the tilted helix. If $\mathbf{Q}' = \mathbf{0}$, so that the c-axis moments are ferromagnetically ordered, the resulting structure is the *cone*.

The magnetoelastic contributions require special treatment when the structures are modulated, because of the limited ability of the lattice to adapt to various strain configurations, when the strains are spatially modulated. The magnetoelastic Hamiltonians considered above are only strictly valid in the uniform case, but they may be generalized to non-uniform structures by replacing the strains by their local values $\epsilon_{\alpha\beta}(i)$, at least in the limit where the wavelength of the modulation is much longer than the range of the interactions. At shorter wavelengths, the form of the magnetoelastic-interaction Hamiltonian may still be applicable, but the effective coupling parameters may depend on the wave-vector. This suggests that the above discussion may be largely unchanged if the magnetic structure is modulated, provided that we take account of the new constraints which we shall now examine. The displacement of the ith ion, $\mathbf{u}(\mathbf{R}_i) = \widetilde{\mathbf{R}}_i - \mathbf{R}_i$, from its equilibrium position $\mathbf{R}_i$ may be divided into a uniform and a non-uniform component, and the non-uniform part may be written as a linear combination of contributions from the *normal phonon modes* at various wave-vectors. It follows from this that a displacement of the ions which varies with a certain wave-vector should be describable in terms of the normal phonon modes at that particular wave-vector, in order to ensure that such a displacement is compatible with the lattice.

To be more specific, we shall consider the wave-vector to be along the c-axis in the hcp lattice. In the *double-zone representation*, which neglects the two different displacements of the hexagonal layers, there are only three normal modes; one longitudinal and two energetically-degenerate transverse modes. All three modes correspond to rigid displacements of the hexagonal layers. The γ-strains describe an elongation of these layers along a certain direction in the plane. If the γ-strains are uniform within each hexagonal layer, the magnitude or the direction of the elongation cannot be allowed to vary from one layer to the next, as this would destroy the crystal. Hence, even though $\langle Q_2^2(\mathbf{J}_i)\rangle$ in the equilibrium equation for $\epsilon_{\gamma 1}(i)$, corresponding to eqn (2.2.25), varies in a well-defined way in a helical structure with $\mathbf{Q}$ along the c-axis, $\epsilon_{\gamma 1}(i)$ is forced to stay constant. The site-dependent version of (2.2.25) is only valid when the right-hand sides are replaced by their averages with respect to any variation along the c-axis, and these averages vanish in the helix. This phenomenon was named the *lattice clamping effect* by Cooper (1967), and further discussed by Evenson and Liu (1969). One of its consequences is that the γ-strain contributions (2.2.27) to the free energy cancel out in the helical phase. This behaviour of the γ-strains therefore enhances the tendency of the wave-vector of the helix to jump to one of the two commensurable values $Q = 0$ or $2\pi/c$, or may increase the stability of other commensurable structures which have a net moment in the basal-plane.

The only strain modes which are allowed to vary along the c-axis are those deriving from the transverse modes, which are $\epsilon_{\varepsilon 1}(i)$ and $\epsilon_{\varepsilon 2}(i)$, and the longitudinal component $\epsilon_{33}(i)$. Like the γ-strains, the α-strains $\epsilon_{11}(i)$ and $\epsilon_{22}(i)$ must remain constant. In the longitudinally polarized phase, the ε-strains are not affected by the ordered moment. The uniform α-strains are determined by the average of $Q_l^0(\mathbf{J}_i)$ and, in addition, the c-axis moments induce a non-uniform longitudinal-strain mode $\epsilon_{33}(i) \propto \langle J_{i\zeta}\rangle^2$ at the wave-vector $2\mathbf{Q}$, twice the ordering wave-vector. The amplitude $\epsilon_{2\mathbf{Q}}$, in $\epsilon_{33}(i) = \epsilon_{2\mathbf{Q}} \cos(2QR_{i\zeta})$, may be determined by the equilibrium conditions for the single sites, with the magnetoelastic-coupling parameters replaced by those corresponding to $2\mathbf{Q}$. The longitudinal strain at site i is directly related to the displacement of the ion along the ζ-axis; $\epsilon_{33}(i) = \partial u_\zeta/\partial R_{i\zeta}$ and hence $u_\zeta(\mathbf{R}_i) = (2Q)^{-1}\epsilon_{2\mathbf{Q}} \sin(2\mathbf{Q}\cdot\mathbf{R}_i)$. Below T_N', where $\langle J_{i\xi}\rangle$ becomes non-zero, the cycloidal ordering induces an $\epsilon_{\varepsilon 1}$-strain, modulated with the wave-vector $2\mathbf{Q}$. The presence of a (static) transverse phonon mode polarized along the ξ-direction corresponds to $\partial u_\xi/\partial R_{i\zeta} = \epsilon_{13}(i) + \omega_{13}(i) \neq 0$, whereas $\partial u_\zeta/\partial R_{i\xi} = \epsilon_{13}(i) - \omega_{13}(i) = 0$. Hence it is $\epsilon_{\varepsilon 1}(i) + \omega_{13}(i)$, with $\omega_{13}(i) = \epsilon_{\varepsilon 1}(i)$, which becomes non-zero, and not just $\epsilon_{13}(i) = \epsilon_{\varepsilon 1}(i)$. In these expressions, $\omega_{13}(i)$ is the antisymmetric part of the strain tensor, which in the long-wavelength limit describes a rigid rotation of the system around the η-axis. Because such a rotation, in the absence of external fields, does not change the energy in this limit, the magnetoelastic Hamiltonian may still be used for determining $\epsilon_\varepsilon(i)$. Only when the relation between the strains and the transverse displacements is considered, is it important to include the antisymmetric part. In helically-ordered systems, the γ-strains are zero, due to the clamping effect, as are the ε-strains, because the moments are perpendicular to the c-axis. Only the α-strains may be non-zero, and because $\langle Q_l^0(\mathbf{J}_i)\rangle$ are independent of the direction of the basal-plane moments, the α-strains are the same as in the ferromagnet (we neglect the possible six-fold modification due to B_α^{66} in (1.4.10)). Their contributions to the axial anisotropy (2.2.33), to be included in κ_l^0, are also the same as in the ferromagnetic case. In the basal-plane ferromagnet, the ε strains contribute to the axial anisotropy $1/\chi_{xx}$ in eqn (2.2.19b):

$$\Delta(1/\chi_{xx}) = \frac{1}{N(\sigma J)^2}\partial^2 F_\varepsilon/\partial\theta^2 = -\tfrac{1}{4}c_\varepsilon H_\varepsilon^2/(\sigma J)^2 \quad ; \quad \theta_0 = \frac{\pi}{2}, \qquad (2.2.34)$$

as derived from (2.2.31). It is straightforward to see that we get the equivalent contribution in the helix at $\mathbf{q} = \mathbf{0}$ in eqn (2.2.33), except that the coupling parameters in (2.2.34) should have the effective values at the wave-vector $\mathbf{Q}$. In the conical phase, both $\epsilon_{\varepsilon 1}(i)$ and $\epsilon_{\varepsilon 2}(i)$ become non-zero, 90° out of phase with each other, corresponding to a transverse displacement of the planes, in a direction which follows the orientation of the moments in the basal plane.