Hamilton–Jacobi teori (i)

Kanonisk transformation fra $(\mathbf{q},\mathbf{p}) = \big(\mathbf{q}(t),\mathbf{p}(t)\big)$ til $(\mathbf{Q},\mathbf{P}) = \Big(\mathbf{Q}(\mathbf{q}_0,\mathbf{p}_0),\mathbf{P}(\mathbf{q}_0,\mathbf{p}_0)\Big)$, hvor $\mathbf{q}_0 = \mathbf{q}(0)$ og $\mathbf{p}_0 = \mathbf{p}(0)$.

$H = H(\mathbf{q},\mathbf{p},t) \mapsto K = K(\mathbf{q}_0,\mathbf{p}_0,t) = 0$ (konstant) $\Rightarrow \dfrac{\partial K}{\partial P_i} = \dot{Q}_i = 0, \quad \dfrac{\partial K}{\partial Q_i} = -\dot{P}_i = 0.$

$$K = 0 \text{ og } K = H + \frac{\partial F}{\partial t} \quad \Rightarrow \quad H(\mathbf{q},\mathbf{p},t) + \frac{\partial F}{\partial t} = 0$$

$$F = F_2(\mathbf{q},\mathbf{P},t) - Q_i P_i \quad \Rightarrow \quad Q_i = \frac{\partial F_2}{\partial P_i}, \quad p_i = \frac{\partial F_2}{\partial q_i}, \quad \frac{\partial F}{\partial t} = \frac{\partial F_2}{\partial t} \qquad \text{eller}$$

Hamilton–Jacobi ligningen: $\qquad H\Big(q_1,\dots,q_n,\dfrac{\partial F_2}{\partial q_1},\dots,\dfrac{\partial F_2}{\partial q_n},t\Big) + \dfrac{\partial F_2}{\partial t} = 0$

Første-ordens partiel differential ligning i $n+1$ variable $q_1,\dots,q_n,t$. Bemærk at ligningen ikke involverer F_2's afhængighed af de n konstante komponenter af $\mathbf{P}$. En løsning til differentialligningen, $F_2 = S =$ *Hamiltons principale funktion* er specificeret ved $n+1$ integrationskonstanter $\alpha_1,\dots,\alpha_{n+1}$, hvoraf én er en triviel additiv konstant:

$F_2 = S(q_1,\dots,q_n,\alpha_1,\dots,\alpha_n,t) + \alpha_{n+1}$ og vælges $\alpha_i = P_i \quad (i = 1,\dots,n)$ fås

$$Q_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial S(\mathbf{q},\boldsymbol{\alpha},t)}{\partial \alpha_i} = \beta_i \Rightarrow q_i = q_i(\boldsymbol{\beta},\boldsymbol{\alpha},t) = q_i\Big(\mathbf{Q}(\mathbf{q}_0,\mathbf{p}_0),\mathbf{P}(\mathbf{q}_0,\mathbf{p}_0),t\Big)$$

$$p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial S(\mathbf{q},\boldsymbol{\alpha},t)}{\partial q_i} = p_i(\mathbf{q},\boldsymbol{\alpha},t) = p_i\big(\mathbf{q}(\boldsymbol{\beta},\boldsymbol{\alpha},t),\boldsymbol{\alpha},t\big).$$

Hamilton–Jacobi teori (ii)

Hamilton–Jacobi ligningen: $H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \Rightarrow \quad S = S(\mathbf{q}, \boldsymbol{\alpha}, t)$

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} = p_i\dot{q}_i + \frac{\partial S}{\partial t} = p_i\dot{q}_i - H = L$$

$$S = \int_{t_1}^{t} L(t')\,dt' \quad \Rightarrow \quad \delta S(t_2) = 0 \quad \Rightarrow \quad \text{Lagrangeligningerne}$$

I tilfældet $\frac{\partial H}{\partial t} = 0$ er H en bevægelseskonstant, $H = E$, og dermed

$$\frac{\partial S}{\partial t} = -E \quad \Rightarrow \quad S(\mathbf{q}, \boldsymbol{\alpha}, t) = W(\mathbf{q}, \boldsymbol{\alpha}) - E\,t \qquad (W\text{: Hamiltons karakteristiske funktion})$$

$$p_i = \frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i} \quad \Rightarrow \quad \frac{dW(\mathbf{q}, \boldsymbol{\alpha})}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i = p_i\dot{q}_i \text{ eller}$$

$$W = \int_{t_1}^{t} p_i\dot{q}_i dt' = \int_{\mathbf{q}(t_1)}^{\mathbf{q}(t)} p_i dq_i \qquad \text{(virkningsintegralet)}$$

Klassisk mekanik $\leftrightarrow$ Kvantemekanik:

$$p_j = \frac{\partial S}{\partial q_j} \quad \leftrightarrow \quad \hat{p}_j = \frac{\hbar}{i}\frac{\partial}{\partial q_j} \;\Rightarrow\; [\hat{q}_j, \hat{p}_k] = i\hbar\,\delta_{jk}$$

$$H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = -\frac{\partial S}{\partial t} \;\leftrightarrow\; H\left(\mathbf{q}, \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{q}}, t\right)\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} \quad \text{(tidsafhængige Schrödingerligning)}$$

$$H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}\right) = E \;\leftrightarrow\; H\left(\mathbf{q}, \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{q}}\right)\psi = E\psi \quad \text{(tidsuafhængige Schrödingerligning)}$$

Den éndimensionale oscillator (i)

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E, \qquad \omega = \sqrt{\frac{k}{m}}, \qquad \frac{\partial H}{\partial t} = 0, \qquad H = T + V = E$$

Hamilton-Jacobi ligning, $S = W - E\,t$: $\qquad H\left(q, \frac{\partial W}{\partial q}\right) = E \quad$ eller

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + (m\omega q)^2\right] = E \quad \Rightarrow \quad W = W(q,\alpha) = W(q,E) = \pm\int\sqrt{2mE - (m\omega q)^2}\,dq$$

E er integrationskonstanten α, udover den “trivielle” additive konstant (α_{n+1}):

$$P = \alpha = E, \qquad Q = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial E} - t = -t \pm \int \frac{m dq}{\sqrt{2mE - (m\omega q)^2}} \qquad \text{eller}$$

$$Q + t = \pm\sqrt{\frac{m}{2E}}\int\frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2E}}} = \pm\frac{1}{\omega}\int\frac{du}{\sqrt{1-u^2}}, \quad u = q\sqrt{\frac{m\omega^2}{2E}} \quad \Rightarrow$$

$$t + Q = \pm\frac{1}{\omega}\arcsin\left(q\sqrt{\frac{m\omega^2}{2E}}\right) \quad \Rightarrow \quad q = \sqrt{\frac{2E}{m\omega^2}}\sin\left(\omega t + \beta\right), \quad Q = \frac{\beta}{\omega}$$

$$p = \frac{\partial S}{\partial q} = \pm\sqrt{2mE - (m\omega q)^2} = \pm\sqrt{2mE}\sqrt{1 - \sin^2(\omega t + \beta)} = \sqrt{2mE}\cos(\omega t + \beta)$$

[$(Q,P) = \left(\beta/\omega, E\right) \mapsto \left(\beta, E/\omega\right)$, hvis vi havde valgt integrationskonstanten $\alpha = P = E/\omega$. Øvelse: Hvad er den ikke-trivielle forskel mellem denne kanoniske transformation og den der blev anvendt i afsnit 9.3?]

Den éndimensionale oscillator (ii)

Indsættes $q_0 = q(0) = \sqrt{\dfrac{2E}{m\omega^2}}\sin\beta, \quad p_0 = p(0) = \sqrt{2Em}\cos\beta$ fås

$$P = P(q_0, p_0) = E = \frac{p_0^2}{2m} + \tfrac{1}{2}m\omega^2 q_0^2, \qquad Q = Q(q_0, p_0) = \frac{\beta}{\omega} = \frac{1}{\omega}\mathrm{Arctan}\left(\frac{m\omega q_0}{p_0}\right)$$

Fortegnene for q og p er et konsistent valg og svarer til at benytte $\pm|\cos(\omega t + \beta)| = \cos(\omega t + \beta)$ i udtrykket for W:

$$W = \pm\int\sqrt{2mE - (m\omega q)^2}\,dq = \pm\int\sqrt{2mE\big[1 - \sin^2(\omega t + \beta)\big]}\sqrt{\frac{2E}{m\omega^2}}d\big[\sin(\omega t + \beta)\big]$$

$$= \frac{2E}{\omega}\int\cos(\omega t + \beta)d\big[\sin(\omega t + \beta)\big] = 2E\int\cos^2(\omega t + \beta)dt \quad \Rightarrow$$

$$S = W - E\,t = 2E\int\big[\cos^2(\omega t + \beta) - \tfrac{1}{2}\big]dt$$

Lagrangefunktionen kan omskrives som følger:

$$L = T - V = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 q^2 = E\cos^2(\omega t + \beta) - E\sin^2(\omega t + \beta) = 2E\big[\cos^2(\omega t + \beta) - \tfrac{1}{2}\big]$$

Dermed ses at resultatet for S er i overensstemmelse med at $S = \displaystyle\int L dt$.

For en oscillator i s dimensioner er $\quad H = \dfrac{1}{2m}\displaystyle\sum_{k=1}^{s}\Big[p_k^2 + (m\omega_k q_k)^2\Big]$ og de variable kan separeres i Hamilton–Jacobi differentialligningen: $W = W_1(q_1, \alpha_1) + W_2(q_2, \alpha_2) + \cdots$

$$\frac{1}{2m}\sum_{k=1}^{s}\left[\left(\frac{\partial W}{\partial q_k}\right)^2 + (m\omega_k q_k)^2\right] = E \quad \Rightarrow \quad \left(\frac{\partial W_k}{\partial q_k}\right)^2 + (m\omega_k q_k)^2 = 2mE_k, \quad E = \sum_k E_k$$