Thermodynamics in finite time: extremals for imperfect heat engines

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A general description is developed for processes involving work and two heat reservoirs or three heat reservoirs in terms of rates for continuous processes or of cycle averages for periodic processes. The description is applied to heat engines having friction, thermal resistance, and heat losses in order to determine the maximum power and maximum efficiency of such engines. By use of a geometric representation the reversible and irreversible parts of a process are separated as the components of a vector. This leads to the definition of a dimensionless quantity that measures irreversibility and is related in a complementary way to the traditional concept of efficiency. The new quantity appears to be useful in cases where efficiency has no well-defined meaning.

I. INTRODUCTION

The first goal of traditional thermodynamics was evaluating how well heat engines perform and how well they might perform in an idealized limit. The idealized limit became the reversible Carnot engine. The corresponding criteria of merit are the capacity of this engine to convert heat at a high temperature into work (the efficiency η) and the ratio of actual work performed to the work that a reversible Carnot engine could perform with the same heat input (the effectiveness). This approach served many useful purposes, but the high degree of idealization of the reversible Carnot engine puts a serious limit on what it can tell us about how well real engines might perform. We would like to have general methods for finding the maximum efficiency and maximum effectiveness for realistic engines operating at finite rates. This series of papers directs itself to finding such extrema.

We previously investigated the optimal operation of a simple model[1] and extended the use of thermodynamic potentials to quasistatic processes.[2] The first approach[1] involved finding the detailed time dependence of the various thermodynamic quantities using optimal control theory. This, of course, is the most complete analysis, but the resulting differential equations are frequently too difficult to solve and in many cases such detailed information is not needed. Thermodynamic potentials[2] offer a more economical way of determining the maximum work that can be extracted from a given process without describing how it is to be carried out. Still, however, one has to solve a differential equation for each particular process.

In the present paper we develop a formalism which focuses on processes of energy conversion, treating the energy conversion system itself as a black box. The approach we use begins much like the method given by Keenan[3] for describing energy availability and work in real and ideal steady-flow systems. Keenan's prescription in its simplest form takes the energy conservation equation for a working fluid, which, with neglect of heat losses and the change of kinetic energy of the fluid, becomes an equation relating the specific enthalpy of the fluid at the entrance to the sum of the specific work and specific enthalpy at the exit. With inclusion of kinetic energy changes and real losses, the formulation permits one to evaluate the work done by real steady-flow and reciprocating systems, their changes in availability, and their effectivenesses and engine efficiencies.

We seek a description of a general heat/work process based, not on the enthalpies of a working fluid at different points as Keenan does, but on the heat flows obtained by the interaction of three heat reservoirs at three different temperatures T_1, T_2, and T_3. This includes the possibility that one is a work reservoir for which $T\equiv\infty$. The process can be represented by a triangular diagram as shown in Fig. 1, which we cavalierly call tricycles. (The choice of the name is discussed below.) The basic idea is that a tricycle cannot itself consume or produce energy, so that conservation

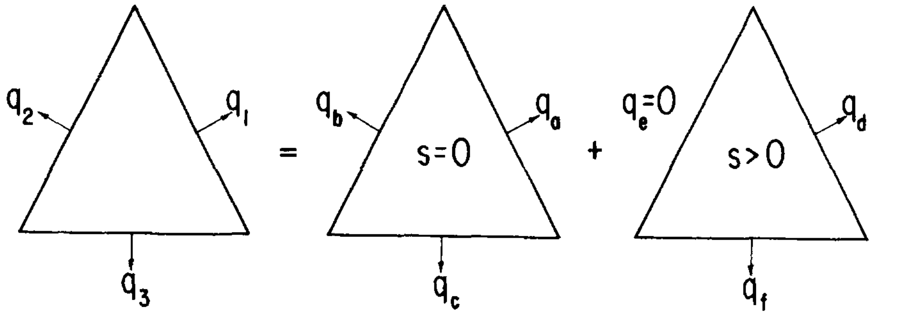

FIG. 1. Decomposition of tricycle (q_1, q_2, q_3) into a reversible part (with zero entropy production) and an irreversible part consisting of heat flow between reservoirs 1 and 3 uncoupled from reservoir 2.

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laws govern the heat flows. These laws, applied to certain classes of tricycles, are used to find the maximum power output and efficiency for realistic engines. The triangular pictures have been helpful in clarifying the decomposition of processes into reversible and irreversible parts. In a later section, a geometric interpretation of the tricycle concept is developed to help quantify the degree of irreversibility of a process.

II. DEFINITION OF A TRICYCLE

A tricycle represents the interaction of three heat reservoirs at temperatures T_1, T_2, and T_3 with the corresponding heat flows q_1, q_2, and q_3 all taken to be positive for heat leaving the system. The rate of heat exchange q_i may either be thought of as an instantaneous value $q_i = dQ_i/dt$ or as an average over a cycle $q_i = Q_i/\Delta t$. The differential form is appropriate to continuous systems such as turbines, and the difference form, to systems that cycle mechanically so that Δt is the period and Q_i is the ith heat flow per cycle. In either case the system returns to its original state, so that

$$q_1 + q_2 + q_3 = 0 \ . \tag{1}$$

In this sense, the energy conversion system, whether continuous or periodic, is cyclic; and therefore its representation as a triangle is given the name "tricycle." The rate of entropy production in the process is, like heat flow, either $s = dS_{\text{prod}}/dt$ or $\Delta S_{\text{prod}}/\Delta t$; and the rate of entropy production—continuous or cycle-averaged—is:

$$s = q_1\tau_1 + q_2\tau_2 + q_3\tau_3 \geq 0 \ , \tag{2}$$

where we define $\tau_i = T_i^{-1}$. A work reservoir has $\tau = 0$ and thus produces no entropy. Most systems of interest will correspond to one τ equal to zero; however, the absorption refrigerator is an example for which all three τ's are nonzero.

Any tricycle fulfilling Eqs. (1) and (2) represents a physically possible process. To further analyze the interaction we want to decompose it into its reversible and (totally) irreversible components. The former is unambiguously defined by $s = 0$, but there is a degree of choice about the meaning of "the amount of irreversibility." One such choice is presented in Fig. 1, where q_1, q_2, and q_3 is decomposed into the reversible q_a, q_b, and q_c, and the irreversible heat flow from τ_1 to τ_3 ($q_e = 0$). Others, in particular one totally symmetric in the three reservoirs, are presented in section 4. Although the temperatures are free to vary within the bounds of Eq. (2), it may be useful to visualize the tricycles with $\tau_1 < \tau_2 < \tau_3$ or equivalently $T_1 > T_2 > T_3$. In that case a tricycle with $q_2 > 0$ is a refrigerator or heat pump, one with $q_2 < 0$ and $\tau_1 = 0$ is a conventional heat engine.

With the decomposition of Fig. 1 it is easy to derive:

$$q_a/(\tau_3 - \tau_2) = q_b/(\tau_1 - \tau_3) = q_c/(\tau_2 - \tau_1) \ ,$$
$$q_d = -q_f = q_1 + q_2(\tau_3 - \tau_2)/(\tau_3 - \tau_1) \ , \tag{3}$$

and

$$s = q_1(\tau_1 - \tau_3) + q_2(\tau_2 - \tau_3) \ . \tag{4}$$

III. EXAMPLES OF WORK-HEAT-HEAT TRICYCLES

Of the three originally independent variables q_1, q_2, and q_3, one may be fixed through Eq. (1), another one by defining a functional form for the losses q_f, and the last degree of freedom will then specify the extent of the process. In this section we look at tricycles where side 1 is a work reservoir, so that $\tau_1 = 0$, and we relabel $q_1 = w$. We will apply different expressions for q_f representing friction, heat leak, and thermal resistance, and derive the maximum power w and efficiency $\eta = w/(-q_2)$ that can be obtained from these tricycles.

A. Friction and heat leak

We define the losses to be

$$q_f = \alpha q_a^2 + q_0 \ , \tag{5}$$

where the coefficient of friction $\alpha > 0$, and q_0 is a rate-independent heat leak from τ_1 to τ_3 which tends to speed up the most efficient operation of the tricycle. The quadratic friction term corresponds to a thick layer of fluid lubricant between sliding surfaces.[4] Other forms apply for thin lubricant layers and for dry surfaces. The power is $w = q_a - q_f$. We set its derivative with respect to q_2 equal to zero to find the maximum power:

$$w = (1/4\alpha) - q_0 \ , \tag{6}$$

obtained for:

$$q_2 = \frac{-1}{2\alpha}\,\frac{\tau_3}{\tau_3 - \tau_2} \ . \tag{7}$$

Maximizing the efficiency gives two solutions:

$$\eta = \frac{\tau_3 - \tau_2}{\tau_3}\,(1 \pm 2\sqrt{\alpha q_0}) \ , \tag{8}$$

for

$$q_2 = \pm\frac{\tau_3}{\tau_3 - \tau_2}\sqrt{\frac{q_0}{\alpha}} \ ,$$

and

$$w = -2q_0 \mp \sqrt{\frac{q_0}{\alpha}} \ , \tag{9}$$

with the upper signs for a heat pump and the lower signs for a heat engine.

B. Thermal resistance

Let us for a moment drop the black box restriction of the tricycle and look inside it. We assume that it is composed of another tricycle—dashed in Fig. 2—connected to the outer reservoirs at τ_2 and τ_3 through thermal resistances ρ_2 and ρ_3, so that its actual operating temperatures are:

$$T_1' = T_1(=\infty) \ ,$$
$$T_2' = T_2 - q_2\rho_2 \ , \tag{10}$$
$$T_3' = T_3 - q_3\rho_3 \ .$$

We now decompose the tricycle into reversible and irreversible *internal* tricycles as in Fig. 2 to find the reversible parts:

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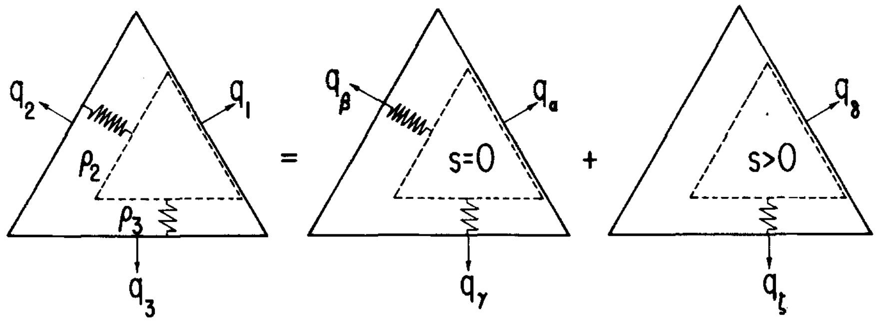

FIG. 2. Tricycle with thermal resistances ρ_2 and ρ_3 between reservoirs 2 and 3 and an interior tricycle (dashed). The interior tricycle is decomposed into its reversible and irreversible parts in analogy to Fig. 1.

$$q_\alpha = -q_2 + T_3 q_2/(T_2 - q_2\rho) ,$$

$$q_\beta = q_2 = q_b , \tag{11}$$

and

$$q_\gamma = -T_3 q_2/(T_2 - q_2\rho) ;$$

and the irreversible parts:

$$q_\delta = -q_\zeta = q_1 + q_2 - T_3 q_2/(T_2 - q_2\rho) ,$$

where,

$$\rho = \rho_2 + \rho_3 . \tag{12}$$

Decomposition of the same outer tricycle according to Fig. 1 and Eq. (3) yields:

$$q_3 = q_c + q_f = q_\gamma + q_\zeta ,$$

or

$$q_f = q_\zeta + (q_a - q_\alpha) ,$$

which means that the "uncaptured work" arising from the thermal resistance is:

$$q_a - q_\alpha = -\frac{T_3}{T_2}\,\frac{q_2^2\rho}{T_2 - q_2\rho} . \tag{13}$$

This quantity must be positive to have physical significance, which requires that:

$$\Delta T \equiv q_2\rho < T_2 . \tag{14}$$

Using the expression (13) for the loss term q_f gives a maximum power output of:

$$w = -\rho^{-1}(\sqrt{T_2} - \sqrt{T_3})^2 , \tag{15}$$

for

$$\Delta T = T_2 - \sqrt{T_2 T_3} . \tag{16}$$

These results are identical to the ones obtained in an analysis[5] of a Carnot engine with finite heat conductance to its reservoirs. However, some care must be exercised in the comparison, since the Carnot engine described by Curzon and Ahlborn only absorbs and discharges heat through the thermal resistances in the respective *half*-periods of its cycle, whereas Eqs. (10) imply that the coarse-grained tricycle assumes constant heat flow, or that ρ_1 and ρ_2 are *effective* values of the conductances that permit the actual heat flows to be represented by an equivalent constant flow. Consequently the values of ρ_2 and ρ_3 must be twice those used in Ref. 3 in order to produce the same losses.

Maximum efficiency is obtained for the value of q_2 for which

$$d\eta/dq_2 = -\rho T_3/(T_2 - q_2\rho)^2$$

equals zero. No such finite, nonzero q_2 exists. The efficiency increases monotonically, as $q_2 \to 0$, just as intuition suggests.

C. Friction, heat leak and thermal resistance

Combining the terms (5) and (13) which together represent the difference between ideal and real work, we obtain an accurate description of real heat machines with their frictional losses, constant heat losses, and resistance to heat transfer. The power, continuous or average depending on the system, is

$$w = -\alpha\left(\frac{T_2 - T_3 - q_2\rho}{T_2 - q_2\rho}\right)^2 q_2^2 - \frac{T_2 - T_3}{T_2}\, q_2 + \frac{T_3}{T_2}\,\frac{q_2^2\rho}{T_2 - q_2\rho} - q_0 . \tag{17}$$

The extrema of Eq. (17) are located at:

$$\Delta T = \begin{cases} T_2 \pm \sqrt{T_2 T_3} , \text{ and} \\ \tfrac{1}{2}[-\nu + T_2 - T_3 \mp \sqrt{(\nu + T_2 + T_3)^2 - 4T_2T_3}] , \end{cases} \tag{18}$$

where

$$\nu = \rho/2\alpha , \tag{19}$$

as substitution of (18) in (17) shows. (Direct derivation of the extrema of (17) is a rather lengthy exercise, not recommended for the casual reader.)

A contour diagram of $-\rho w(\nu, \Delta T)$ for $T_2 = 9$, $T_3 = 1$ and $\Delta T_0 \equiv q_0\rho = 0$ is shown in Fig. 3 with the curves of maxima and minima drawn in heavy solid and dashed lines respectively. The physical region, $\Delta T < T_2$, is divided into a frictionally dominated region for $\nu > -(\sqrt{T_2} - \sqrt{T_3})^2$ with two maxima and a minimum, and a resistively dominated region for $\nu < -(\sqrt{T_2} - \sqrt{T_3})^2$ with one maximum. Furthermore, the two maxima have the same value. Calling the four roots (18) $\Delta T_{a,b,c,d}$ we have ΔT_a always nonphysical, and

$$-\rho w(\Delta T_c) = -\rho w(\Delta T_d) = \Delta T_0 - \tfrac{1}{2}\nu , \tag{20}$$

so the same extremal power output may be obtained with two different expenditures of heat input and thus at different efficiencies. The extremum $-\rho w(\Delta T_b)$ (and thus w) is a monotonically decreasing function of ν, so no global maximum of w exists for finite α.

The region around the point where the three roots $\Delta T_{b,c,d}$ coalesce is very flat, viz., to second order in

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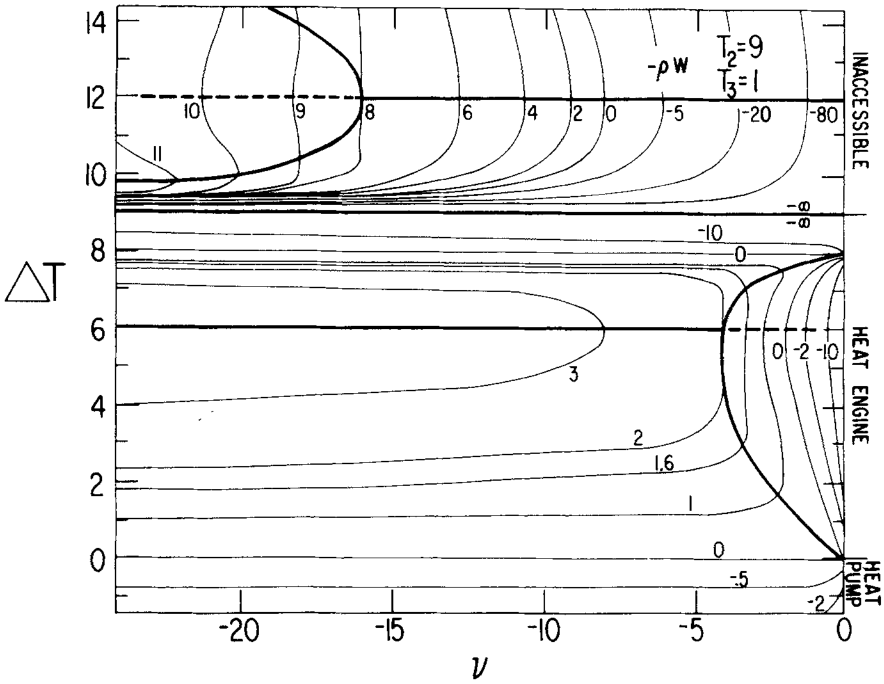

FIG. 3. Contour map of $-\rho w$ of (17) as a function of relative friction ($\nu = \rho/2\alpha$) and heat exchange rate with reservoir 2 ($\Delta T = \rho q_2$) for a work-heat-heat tricycle with friction and thermal resistance. The figure is drawn for reservoir temperatures $T_2 = 9$ and $T_3 = 1$, in arbitrary units. The heavy lines indicate singularities and maxima (full line) and minima (broken line), Eq. (18). The region of ν near zero is the frictionally-dominated region with two real and equal maxima, and the region of larger negative ν is the resistance-dominated region with only a single maximum.

variations around the bifurcation point $(\nu, \Delta T)_* = [-(\sqrt{T_2} - \sqrt{T_3})^2,\ T_2 - \sqrt{T_2 T_3}]$,

$$-\rho w = \frac{z^2}{2}\left(1 + \frac{v}{z^2} - \frac{v^2}{z^4}\right) + \Delta T_0 - \frac{u^2}{\sqrt{T_2 T_3}}\,\frac{v}{z^2}\left(1 - \frac{v}{z^2}\right)\ , \tag{21}$$

with

$$(\nu, \Delta T) = (\nu, \Delta T)_* + (v, u)\ ,$$

and

$$z = \sqrt{T_2} - \sqrt{T_3}\ .$$

In order to give the dimensionless quantities in Fig. 3 more physical meaning let us make each temperature "unit" equal to 300 °K, so $T_2 = 2700$ °K and $T_3 = 300$ °K. At the point $(\nu, \Delta T)_*$ a thermal resistance of -1°/kW, a reasonable value for heat machines of the 100 kW class like car engines, and frictional coefficient $\alpha = 4.167 \times 10^{-4}$ kW^{-1} produces a power output $w = 600$ kW at a heat consumption of $q_2 = -1800$ kW, and therefore at the efficiency $\eta = 0.33$. Twice as large friction, $\alpha = 8.333 \times 10^{-4}$ kW^{-1}, makes the machine operate in the frictionally dominated region where the maximum power output $w = 300$ kW can be obtained at either of the heat consumptions $q_2 = -706$ kW or $q_2 = -2294$ kW with corresponding efficiencies $\eta = 0.42$ and 0.13. In the thermal resistance dominated region, where only one maximum exists, a frictional coefficient $\alpha = 2.083 \times 10^{-4}$ kW^{-1} allows a maximum $w = 1200$ kW at $q_2 = -1800$ kW and $\eta = 0.67$.

Whereas finding the extrema (18) of the power output involves solving a quartic equation, the extrema of the efficiency are located at the roots of the quintic:

$$\Delta T^5 - 3T_2 \Delta T^4 - [2\nu(T_3 + \Delta T_0) - (3T_2^2 - 2T_2T_3 - \Delta T_3^2)]\Delta T^3 + T_2[2\nu(T_3 + 3\Delta T_0) - (T_2 - T_3)^2]\Delta T^2 - 6\nu T_2^2 \Delta T_0 \Delta T + 2\nu T_2^3 \Delta T_0 = 0\ , \tag{22}$$

which we have been unable to solve analytically. Instead contour diagrams of $\eta(v, \Delta T) = w/(-q_2)$ are plotted in Fig. 4 for $T_2 = 9$, $T_3 = 1$, and $\Delta T_0 = 0$ and -1. For heat engines ($\Delta T > 0$), small losses correspond to large η ($\eta \to 1$) and for heat pumps ($\Delta T < 0$), to numerically small η ($\eta \to +0$). (The coefficient of refrigerator performance[6] is $q_3/q_1 = \eta^{-1} - 1$.) When $\Delta T_0 = 0$, there are no losses which encourage a fast process, and the best efficiency is $\eta = \eta^{rev} = 0.89$ for an infinitely slow process at $\Delta T = 0$. When there are heat leaks, $\Delta T_0 \neq 0$ creates a singularity at $\Delta T = 0$, and the heat engine region of the contour map is very similar to what was found in the power contour plot with two maxima and a minimum discernible in the frictionally dominated region $\nu \gtrsim -4$ and only one maximum in the resistive region $\nu \lesssim -4$. Heat pumps display only one minimum over the entire interval, which makes $(\eta^{-1} - 1)$ a maximum.

IV. GEOMETRIC INTERPRETATION

Mathematically, a tricycle is an ordered triple of real numbers $q = (q_1, q_2, q_3)$, satisfying the equations

$$\sum q_i = 0, \text{ and } \sum \tau_i q_i \geqq 0\ .$$

Such ordered triples lend themselves naturally to geometric interpretation in which each point of a 3-dimensional q-space corresponds to a process. The geometric picture in turn suggests new ways to examine the physics it describes. One of these is a way to describe the irreversibility of a process, as we shall see, which requires that we examine the concept of distance in q-space. Moreover geometrization is suggestive of ways to treat systems with several reservoirs or even a continuously varying heat bath temperature, such as Rankine or Otto cycles, which we hope to

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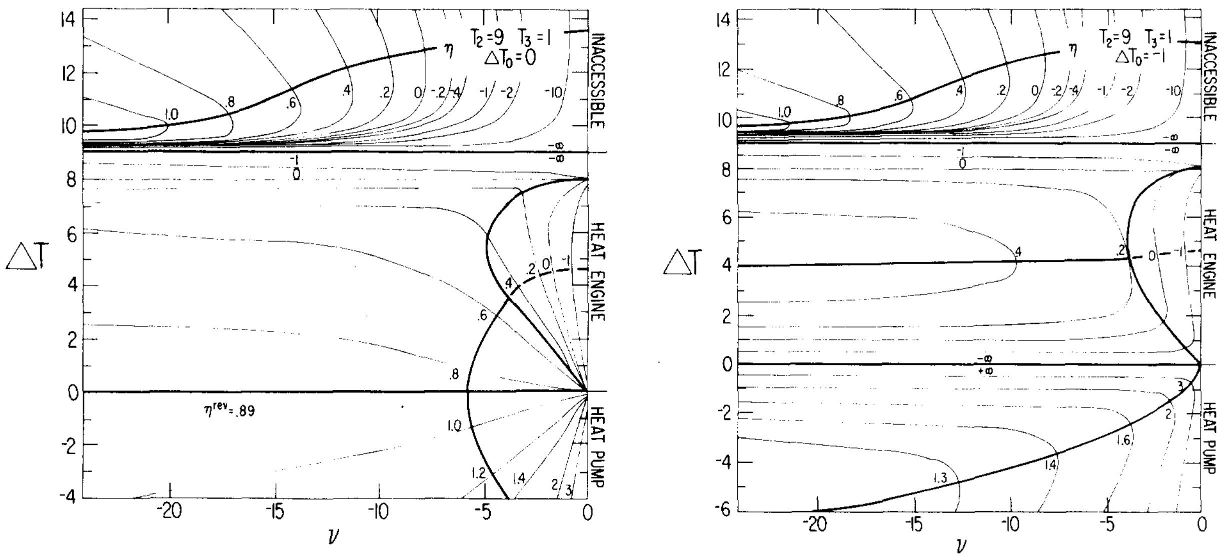

FIG. 4. Contour maps of the efficiency η as a function of relative friction ($\nu=\rho/2\alpha$) and heat exchange rate with reservoir 2 ($\Delta T=\rho q_2$) for a work-heat-heat tricycle with friction, thermal resistance, and constant heat leak $q_0=\Delta T_0/\rho$: (a) $\Delta T_0=0$, so the maximum efficiency is that of the reversible Carnot engine; (b) $\Delta T_0=-1$, which yields contours much like those of Fig. 3. However, unlike the extremal work, the extremal efficiency has a fifth root in the heat pump region of ΔT. The heavy lines indicate singularities and maxima (full line) and minima (broken line). The extrema drawn are estimated solutions of Eq. (22). The reservoir temperatures are the same as for Fig. 3, $T_2=9$, $T_3=1$.

examine in the future. Meanwhile, we concentrate on the 3-dimensional (q_1, q_2, q_3) space and 2-reservoir heat/work or 3-reservoir heat processes.

In order to define distance in this space, one must first adopt a metric.[7] Nothing in the mathematics of this space defines a natural metric, but the physics makes some choices particularly useful. In general, the metric c, restricted here to be diagonal, appears in the scalar product of vectors a and b, thus:

$$\mathbf{a}\cdot\mathbf{b}=\sum_i \frac{a_i b_i}{c_i}, \qquad (23)$$

where c is undetermined for the time being. The length of a vector is as usual:

$$a=|\mathbf{a}|=(\mathbf{a}\cdot\mathbf{a})^{1/2}, \qquad (24)$$

and $\perp$ denotes an orthogonal vector $\mathbf{a}^{\perp}\perp\mathbf{a}$.

The energy conservation Eq. (1) defines a plane which, for convenience, we call the *q-plane*. Similarly, the locus of zero entropy production is the *s-plane* for which Eq. (2) is an equality. The representative point of a reversible process must satisfy both $\sum q_i=0$ and $s=\sum\tau_i q_i=0$, and thus lie on the intersection of the two planes which define a *reversible line* r with unit direction vector:

$$\mathbf{d}_r=\boldsymbol{\theta}/\theta, \qquad (25)$$

where:

$$\boldsymbol{\theta}=(\tau_3-\tau_2,\ \tau_1-\tau_3,\ \tau_2-\tau_1), \qquad (26)$$

with components equal to differences of inverse temperatures. Since a spontaneous process has a positive entropy production, all physical tricycles will be represented by points on that half of the q-plane lying above the s-plane (see Fig. 5).

Decomposition of q into its reversible and irreversible components:

$$\mathbf{q}=\mathbf{q}_r+\mathbf{q}_r^{\perp}, \qquad (27)$$

gives:

$$\mathbf{q}_r=(\mathbf{q}_r\cdot\mathbf{d}_r)\,\mathbf{d}_r=\sum_{i=1}^{3}\left(\frac{q_i\theta_i}{\theta_i}\right)\theta^{-2}\boldsymbol{\theta}, \qquad (28a)$$

$$\mathbf{q}_r^{\perp}=\mathbf{q}-\left[\sum\left(\frac{q_i\theta_i}{c_i}\right)\right]\theta^{-2}\boldsymbol{\theta}, \qquad (28b)$$

which, of course, depend on the metric chosen. A useful dimensionless measure of the irreversibility of the process is the quantity we call the *drive*:

$$\delta=\frac{q_r^{\perp}}{q}=[1-(\mathbf{q}\cdot\boldsymbol{\theta})^2/q^2\theta^2]^{1/2},$$

$$=\left[1-\frac{(\sum q_i\theta_i/c_i)^2}{(\sum q_i^2/c_i)(\sum\theta_i^2/c_i)}\right]^{1/2}, \qquad (29)$$

which is the length of the irreversible part of q, relative to the total length of q. The drive is zero for a reversible process, and unity for a process containing no reversible component $\mathbf{q}_r$. The length of $\mathbf{q}$ and its components depend on the metric, so the vector $\mathbf{q}_r^{\perp}$ does also. Hence there is no invariant (metric-independent) meaning to "totally irreversible." However, we shall see that the extrema of δ for the friction-heat leak-resistance problem occur at the values of ΔT that solve (22), which themselve are invariant.

Any metric of the form:

$$\theta_1/c_1=\theta_3/c_3\neq\theta_2/c_2, \qquad (30)$$

yields:

$$\mathbf{q}_r^{\perp}=\mathbf{q}-(q_2/\theta_2)\boldsymbol{\theta}=\left(q_1-q_2\frac{\theta_1}{\theta_2},\ 0,\ q_3-q_2\frac{\theta_3}{\theta_2}\right), \qquad (31)$$

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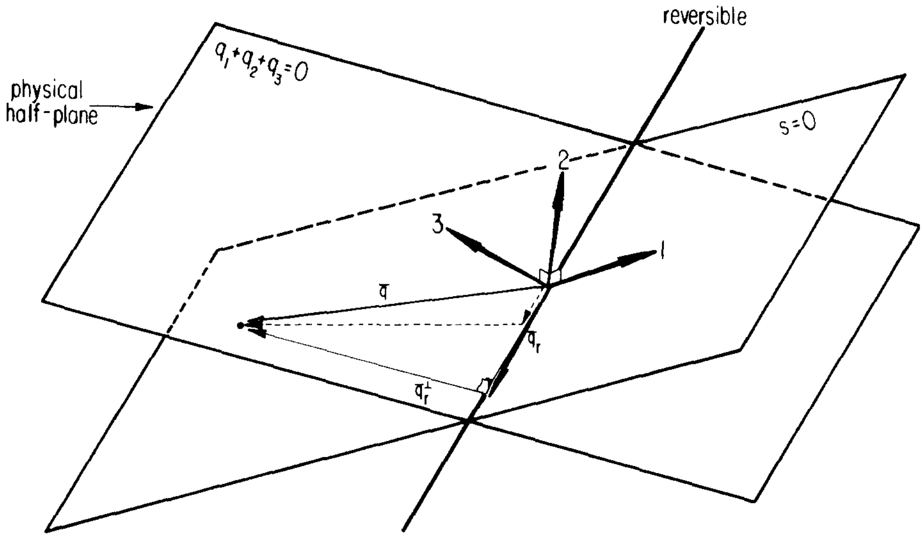

FIG. 5. The (q_1, q_2, q_3) space, with the planes defined by conservation of energy and zero entropy production. Real processes lie on the "physical halfplane" indicated, with the reversible limit given by the intersection of the two planes. Two decompositions of a general tricycle **q** into its reversible and irreversible components are shown. The drive is defined as the ratio of the length of the irreversible component $\mathbf{q}_r^\perp$ to the total length of **q**, the vector representing the real process.

which is identical to the decomposition arbitrarily chosen in Fig. 1. This is shown as the dotted vectors in Fig. 5. The inequality of θ_2/c_2 in Eq. (30) is necessary to keep $\mathbf{q}_r$ from vanishing. A more desirable decomposition would be one that does not single out one component, such as q_2 above, but treats all heat exchanges on an equal footing. One such metric is:

$$\mathbf{c} = C(\theta_1^2, \theta_2^2, \theta_3^2) , \qquad (32)$$

with:

$$C = \left(\sum \theta_i^2\right) \Big/ 3\pi\theta_i^2 . \qquad (33)$$

In this metric the reversible part of the vector **q** has length

$$q_r = \frac{|\pi\theta_i|}{\sqrt{\sum\theta_i^2}} \sum (q_i/\theta_i) , \qquad (34a)$$

while the irreversible part and the drive take on the simple forms:

$$q_r^\perp = s , \qquad (34b)$$

and

$$\delta = s/q . \qquad (34c)$$

This metric measures distance in entropy units.

This 3-dimensional representation (actually 2-dimensional since **q** must be on the physical half plane) of any tricycle is a convenient way of visualizing a heat process, with the drive as a generalized inefficiency.

V. APPLICATION OF METRIC

The analysis of power and efficiency for work-heat-heat tricycles in section 2 was independent of metric. In this section we will use the geometric representation to elaborate on these examples. We include an unspecified uncaptured work f so that:

$$w = -\frac{\tau_3 - \tau_2}{\tau_3} q_2 - f , \quad s = f\tau_3 . \qquad (35)$$

Going back to the general metric for a moment we find from Eq. (28) that:

$$q_r^\perp = s \Big/ \left[\sum \tau_i^2 c_i - \frac{(\sum \tau_i c_i)^2}{\sum c_i}\right]^{1/2} , \qquad (36a)$$

$$q_r = \frac{-1}{\theta\tau_3}\left[q_2\left(\sum \frac{\theta_i^2}{c_i}\right) + s\left(\frac{\theta_1}{c_1} - \frac{\theta_3}{c_3}\right)\right] , \qquad (36b)$$

which says that for all metrics $q_r^\perp$ is proportional to s and q_r is linear in s, so for increasing f the point **q** will describe a straight half-line from the reversible line back towards the heat pump region. Metric (32) is seen to be exactly the one which simplifies Eqs. (36) to:

$$q_r^\perp = s , \qquad (37a)$$

$$q_r = -\frac{\mathrm{sign}(\tau_3 - \tau_2)}{\sqrt{\sum\theta_i^2}} [3q_2\tau_2(\tau_3 - \tau_2) + s(\tau_3 - 2\tau_2)] , \qquad (37b)$$

$$q^2 = \frac{3}{\theta_1^2 + \tau_2^2}\left[\left(\sum\theta_i^2\right)^{-1}[q_2\theta_1\tau_2(\tau_3 - 2\tau_2) - s(\theta_1^2 + \tau_2^2)]^2 + q_2^2\theta_1^2\tau_2^2\right\} . \qquad (37c)$$

Here, as well as in Eqs. (36), the loss term enters only through its accompanying entropy production.

The most nearly reversible process is obtained when the drive δ, Eq. (29), is at its minimum which happens when:

$$\frac{d\delta}{dq_2} = \frac{(\sum\theta_i^2/c_i)^2}{q^3\theta^2\tau_3^2[\sum\tau_i^2 c_i - (\sum\tau_i c_i)^2/\sum c_i]^{1/2}}\left(\frac{ds}{dq_2} q_2 - s\right) \times\left[q_2\sum\theta_i^2/c_i + s\left(\frac{\theta_1}{c_1} - \frac{\theta_3}{c_3}\right)\right] = 0 , \qquad (38)$$

which is satisfied when:

$$\frac{s}{q_2} = \frac{ds}{dq_2} , \qquad (39a)$$

or

$$\frac{s}{q_2} = -\frac{\theta_1^2/c_1 + \tau_3^2/c_2 + \tau_2^2/c_3}{\theta_1/c_1 - \tau_2/c_3} . \qquad (39b)$$

Introducing the full friction, heat leak and thermal resistance losses from Eq. (17), Eq. (39a) becomes identical to Eq. (22), the quintic equation defining the optimal efficiency in section C. Minimization of the

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drive is thus equivalent to maximizing the efficiency η, irrespective of metric. The relation between δ and η for an arbitrary metric is:

$$1-\delta^2 = \frac{\{\eta[c_1^{-1}\tau_3 - \tau_2(c_1^{-1}+c_3^{-1})] + c_2^{-1}\tau_3 + c_3^{-1}\tau_2\}^2}{[c_1^{-1}\eta^2 + c_2^{-1} + c_3^{-1}(1-\eta)^2][c_1^{-1}(\tau_3-\tau_2)^2 + c_2^{-1}\tau_3^2 + c_3^{-1}\tau_2^2]} \quad (40)$$

For metric of Eq. (32), this simplifies somewhat to:

$$1-\delta^2 = \frac{\{\eta[\tau_3-\tau_2)^{-1} - \tau_2^{-1}] + \tau_2^{-1} + \tau_3^{-1}\}^2}{3[\eta^2(\tau_3-\tau_2)^{-2} + \tau_3^{-2} + (1-\eta)^2\tau_2^{-2}]} \quad (41)$$

When $\eta \to \eta^{\text{rev}}$, $\delta \to 0$ independent of metric, but when $\eta \to 0$, δ approaches a value that depends on the metric.

As has been shown in this section, the geometric representation of tricycles is a powerful tool for obtaining information about a general class of heat processes, especially when the equations are not simplified by the presence of a work reservoir. In that case the drive offers a natural way to extend the traditional concept of efficiency in a complementary way closely related to the entropy production.

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