Optimal piston trajectories for adiabatic processes in the presence of friction

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Optimal piston motion is derived for adiabatic processes in the presence of different types of friction and with frictional energy dissipated externally as well as internally. The assumed model for the frictional dissipation affects the optimal motion. We consider a piston fitted in a cylinder containing an ideal gas as the working fluid. The gas starts from a given initial temperature and is assumed to complete the process in a prescribed finite time. The effect of the working gas temperature (or equivalently internal energy) on the viscosity leads to a temperature dependence of the frictional losses. The optimal piston motion then is no longer constant speed, as was concluded previously for the externally dissipative case. In the current study two objectives are considered: maximizing power output and minimizing frictional losses. It is shown that in the externally dissipative mode optimizing power or frictional losses are equivalent, while in the internally dissipative mode the optimal motions are different. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2401313]

I. INTRODUCTION

Classical thermodynamics places bounds on thermodynamic measures based on reversible assumptions.[1] The resulting bounds have limited practical value because reversible operation implies zero rate of operation or infinite system size. Finite-time thermodynamics (FTT) extends the thermodynamic analysis to include the most important rate constraints, e.g., finite heat transfer rates, heat leaks, and friction, while operating within a finite process/cycle time.[2] These methods derive more realistic bounds on performance, often including the paths for achieving such bounds. The methods of FTT have been applied to a wide range of thermodynamic systems.[2–19,22–28] This includes chemical reactions,[28] distillation,[27] and systems attached to a large number of sources.[26]

Heat engines have been treated extensively, in particular, a model heat engine which operates between two heat reservoirs, a hot reservoir at temperature T_H and a cold reservoir at temperature T_C, under the assumption of the Newtonian heat transfer law, the so-called Novikov-Curzon-Ahlborn engine which initiated the whole field. For this engine the efficiency at maximum power production was found to be given by the simple expression[3,4]

$$\eta_{CA} = 1 - \sqrt{T_C/T_H}. \tag{1}$$

Heat pumps were also originally treated with finite heat transfer rate as the sole dissipation mechanism.[5,16,17] Subsequently, reciprocating chillers have been analyzed and their characteristics compared to experimental data.[18,23] The effect of heat leaks and friction on the performance of heat engines

were addressed by Refs. 5, 12, and 13, and later on the performance of heat pumps was discussed in Refs. 14 and 15.

Work on finite-time thermodynamics has followed two main approaches: generic models,[2–8,12–18] and more detailed optimization. The ideal Otto engine,[9,10] the ideal Diesel engine,[11] and the Stirling engine[19] are examples of the latter. In Refs. 9–11, corresponding to the well lubricated sliding motion of solids, the frictional losses were assumed to be proportional to the square of the speed of the piston. For the intake and exhaust strokes in engines fluid friction is an additional important loss mechanism. Its effect was assumed to be similar to the sliding friction and to have the same model dependence on the piston speed; this assumption holds, e.g., in the case of laminar flow. The effect of turbulence and its consequence on the frictional model was studied in detail in Ref. 19. In case of turbulent flow the frictional dissipation (FD) is proportional to the power m of the piston speed (V) ($FD\alpha V^m$), where m is in the range of 2–3.

In the present study we focus on the effect of friction as we account for turbulence and temperature dependence of viscosity of the ideal working fluid. We seek the optimal motion of a piston to complete a process in finite time, and the optimal motion is compared to a sinusoidal motion, a good approximation to the conventional piston motion in practical engines. Two modeling assumptions are compared: externally dissipative friction and internally dissipative friction. Only under special conditions is the optimal path a constant velocity. We also find that the optimal path depends on where the frictional heat is disposed of.

The optimal motions of the Otto engine,[9,10] the Diesel engine,[11] and the Stirling engine[19] were derived following two steps: (i) The individual branches or strokes (compression, expansion, and heat addition/removal) were optimized

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separately and then (ii) pieced together to maximize the net work production per cycle. In the original studies the effect of finite piston acceleration was also analyzed, leading to the conclusion that the maximal improvement in efficiency is 1% when the acceleration is limited to the maximum acceleration of the sinusoidal (conventional) motion, but 15% when there is no acceleration limitation.

In the present study we do not limit acceleration and we consider the case of infinite acceleration so that the derived results yield the upper bound of the improvements. We focus on the optimization of the individual strokes taking into account the effect of friction and its dependence on the temperature of the working fluid. For a full cycle optimization the methods described previously may be applied.[9–11,19]

The article is arranged as follows: In Sec. II the methodology and model are introduced. In Sec. III we derive and solve the equation for optimal motion for maximum power production. In Sec. IV we treat the frictional dissipation term. In Sec. V we derive the optimal motion for minimal dissipation. Finally the conclusions are given in Sec. VI.

II. METHODOLOGY AND MODEL

We consider the following problem: A piston is fitted in a cylinder which contains an ideal gas at initial temperature T_i (or, equivalently, internal energy E_i). The piston is required to move from a given initial volume V_i to a given final volume V_f in a specified time τ. The final internal energy is not specified but depends on the complete piston motion (free-state boundary condition). Dissipation in the system is caused by frictional losses due to mechanical (sliding) friction and/or due to fluid friction. The fluid friction depends on the viscosity of the working fluid, which in turn depends on temperature (internal energy) to some power n.[20] The objective in the current analysis will be to maximize power output or minimize frictional losses. The resulting optimal motions are sensitive to whether the frictional heat is disposed of inside or outside the working fluid. It turns out that maximizing power or minimizing frictional losses for externally dissipative friction are the same under the modeling assumptions considered, whereas the optimal motions are different in the case of internally dissipative friction.

We use optimal control methods or calculus of variations[21] to optimize the specified function under energy conservation. For an ideal gas the power produced is given as the difference between the reversible value and the dissipation,

$$P = \frac{R}{C_v}\frac{E}{V}\dot{V} - \alpha E^n \dot{V}^m, \quad (2)$$

where R is the ideal gas constant, C_v the heat capacity at constant volume of the working fluid, E the internal energy, α the frictional coefficient, n the power of temperature (energy) on which the viscosity depends, and m the power dependence on the friction speed $\dot{V}$. The internal energy of the working fluid balances the power production according to

$$\dot{E} = -\frac{R}{C_v}\frac{E}{V}\dot{V} + \beta \alpha E^n \dot{V}^m, \quad (3)$$

where β is the fraction of the heat of friction which stays inside the engine. A similar division of energy into reservoirs of different temperatures was treated in Ref. 22. In order to consider the two objective functions, power and frictional losses, simultaneously, we introduce the switching parameter δ in the Lagrangian L as follows:

$$L = \frac{R}{C_v}\frac{E}{V}\dot{V}(\delta + \lambda) + \lambda\dot{E} - \alpha E^n \dot{V}^m(1 + \beta\lambda), \quad (4)$$

where $\delta = 0$ corresponds to minimizing frictional losses, while $\delta = 1$ corresponds to maximizing power.

The optimal motion is found by solving the Euler-Lagrange equations,

$$\frac{\partial L}{\partial E} - \frac{d}{dt}\frac{\partial L}{\partial \dot{E}} = 0, \quad (5)$$

$$\frac{\partial L}{\partial V} - \frac{d}{dt}\frac{dL}{\partial \dot{V}} = 0, \quad (6)$$

leading to

$$\dot{\lambda} = \frac{R}{C_v}\frac{\dot{V}}{V}(\delta + \lambda) - \alpha n E^{n-1}\dot{V}^m(1 + \beta\lambda) \quad (7)$$

and

$$\ddot{V} = \frac{R}{mC_v}\frac{\dot{V}^2}{V}\left(n - \beta\frac{\delta + \lambda}{1 + \beta\lambda}\right). \quad (8)$$

The solution of this system of equations, along with the specified boundary conditions including the transversality condition $\lambda_f = 0$, leads to the optimal piston motion. In the following section we determine this optimal motion and consider different special cases.

III. OPTIMAL MOTION FOR MAXIMUM POWER PRODUCTION

In this section we derive the optimal time dependent piston motion for maximizing power ($\delta = 1$). For comparison purposes we consider two extreme values of β, completely externally dissipative friction, $\beta = 0$, where all the heat of friction is lost to the environment, and completely internally dissipative friction, $\beta = 1$, where all the heat of friction is recycled into the working gas. For those two extreme cases the optimal motion is derived by solving the differential equations (7) and (8). We introduce the parameter d,

$$d = \begin{cases} \dfrac{Rn}{mC_v} & \text{for } \beta = 0 \\[2ex] \dfrac{R(n-1)}{mC_v} & \text{for } \beta = 1, \end{cases} \quad (9)$$

in order to determine the optimal motion in both limits simultaneously.
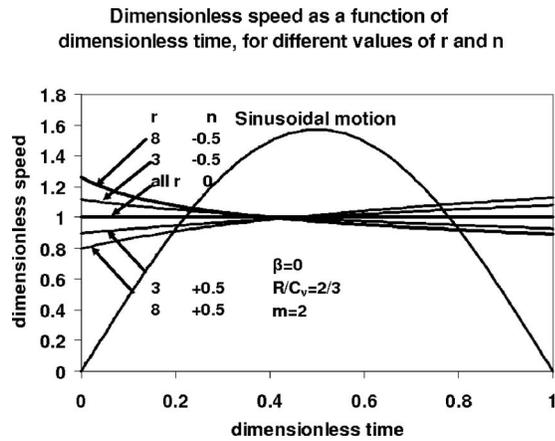
FIG. 1. Dimensionless optimal piston motion $\dot{\nu}^* = \dot{V}_i^* \tau / V_i(r-1)$ vs dimensionless time $t^* = t/\tau$ in the case of externally dissipative friction. The optimal motion is the same for maximizing power and minimizing frictional losses. The optimal piston motion is shown for different values of $n = -0.5$, 0, and 0.5, $r = 3$ and 8, and $m = 2$. The constant speed is optimal only in the case of $n = 0$.

FIG. 2. Dimensionless optimal piston motion $\dot{\nu}^* = \dot{V}_i^* \tau / V_i(r-1)$ vs dimensionless time $t^* = t/\tau$ in the case of internally dissipative friction. The optimal motion is for the case of maximizing power. The optimal motion is shown for different values of $n = -0.5$, 0, and 0.5 and $m = 2.5$. The constant speed is optimal only in the two special cases with $n = 1$ and for minimal losses with $n = 0$.

Without limiting piston acceleration, the optimal motion is given by the following expression for the volume dependence of the gas on time:

$$V^*(t) = V_i\left[1 + (r^{1-d} - 1)\frac{t}{\tau}\right]^{1/(1-d)}, \tag{10}$$

where $r = V_f/V_i$ is the compression ratio. The superscript asterisk indicates an optimal quantity. The optimal time dependent piston speed is thus

$$\dot{V}^*(t) = \dot{V}_i\left[1 + (r^{1-d} - 1)\frac{t}{\tau}\right]^{d/(1-d)}, \tag{11}$$

where $\dot{V}_i$ is the initial speed required to fulfill the final boundary condition $V(t=\tau) = V_f$; it is given by

$$\dot{V}_i = \frac{V_i}{\tau}\frac{r^{1-d} - 1}{1-d}. \tag{12}$$

It is worth noting that the optimal solution for the externally dissipative case ($\beta = 0$), where none of the energy is recycled and with no temperature dependence of the viscosity ($n = 0$) (Ref. [5]), implies a constant process speed. The same conclusion was arrived at for some specific engines[9–11,19] as well as for linear processes in general.[5] Including a temperature dependence of viscosity, the optimal motion depends on n and its sign. The optimal piston speed may increase or decrease with time for different values of n.

In Fig. 1 we plot the optimal dimensionless speed $\dot{\nu}^* = \dot{V}_i^* \tau / V_i(r-1)$ for an externally dissipative friction ($\beta = 0$) as a function of dimensionless time t/τ for a range of exponents n. Along with the optimal motion the sinusoidal is shown for comparison. The optimal piston motion is no longer constant speed, and its behavior depends on the value of n and the compression ratio r. Plots are shown for two compression ratios: $r = 3$, a typical value in Stirling engines, and $r = 8$ for a spark ignition (SI) engine (Otto). In Fig. 2 a similar plot is shown for the externally dissipative case $\beta = 1$.

IV. FRICTIONAL DISSIPATION TERM

In order to find the frictional dissipation term, we need to substitute the time dependent motion into the power and energy expressions. First the time dependence of the internal energy of the working gas is derived from Eq. (3), and then substituting it into the power expression, Eq. (2), we find the total work which includes the reversible term and the dissipation term (friction).

The energy equation is of the Bernoulli type, a nonlinear first order differential equation. By a suitable substitution, $u = E^{1-n}$, it becomes a linear first-order differential equation with the general solution for optimal motion,

$$E^*(t) = E_i\left(\frac{V}{V_i}\right)^{-R/C_v}\left[1 + \beta\alpha(1-n)\right.$$
$$\left.\times V_i\dot{V}_i^{m-1}\frac{(V/V_i)^f - 1}{f}\right]^{1/(1-n)}, \tag{13}$$

with the new constant f defined as

$$f = 1 + \frac{R}{C_v}(\beta - n) + (m-1)d. \tag{14}$$

Integration of the power equation (2) leads to the following expression for work delivered in this process:

$$w^* = E_i - E_f - (1 - \beta)\alpha\int_0^\tau E^n \dot{V}^* dt. \tag{15}$$

Next we derive the frictional dissipation term resulting from the different motion types.

A. Externally dissipative friction

In this case we substitute $\beta = 0$ into the optimal energy and work expressions (13) and (15) to arrive at

$$E^*(t) = E_i \left(\frac{V}{V_i} \right)^{-R/C_\nu}. \tag{16}$$

Note that this is the classical result for an adiabatic process and as such is independent of the actual path followed. The corresponding maximal work produced is then

$$w^* = E_i - E_f - \alpha E_i^n \frac{V_i^m}{\tau^{m-1}} \left(\frac{r^{1-d} - 1}{1 - d} \right)^{m-1} \frac{r^f - 1}{f}. \tag{17}$$

For an imposed sinusoidal motion the internal energy is still given by Eq. (16), whereas the work is

$$w_{\sin} = E_i - E_f - \alpha E_i^n \frac{V_i^m}{\tau^{m-1}} \pi^{m-1} (r-1)^m r^{-nR/C_\nu} B$$

$$\times \left(\frac{m+1}{2}, \frac{m+1}{2} \right) {}^2F_1 \left(\frac{m+1}{2}, \frac{nR}{C_\nu}, m+1; \frac{r-1}{r} \right), \tag{18}$$

where B is the beta function and ${}^2F_i$ is the hypergeometric function.

From these derived formulas we define g_e as the ratio between the frictional dissipation of the sinusoidal motion relative to the optimal motion,

$$g_e = \frac{\pi^{m-1}(r-1)^m r^{-nR/C_\nu} B[(m+1)/2, (m+1)/2] {}^2F_1 \left[(m+1)/2, \frac{nR}{C_\nu}, m+1; (r-1)/r \right]}{[(r^{1-d} - 1)/(1-d)]^{m-1}(r^f - 1)/f}. \tag{19}$$

For temperature independent friction, $n=0$, this expression is reduced to[18]

$$g_e = \pi^{m-1} B \left(\frac{m+1}{2}, \frac{m+1}{2} \right). \tag{20}$$

Expressions (19) and (20) provide quantitative estimates of the potential improvement of the optimal motion compared to the conventional sinusoidal motion. The effects on viscosity from turbulent flow and varying temperature of the working fluid are explicit in Eq. (19). For any n value different from zero, g_e is a function of the compression ratio r, while for $n=0$ the compression ratio dependence disappears and g_e is simplified to Eq. (20). For example, using Eq. (20) with different values of $m=2.0$, 2.5, and 3.0, the frictional losses for sinusoidal motion are 1.23, 1.41, and 1.64 times as large as the frictional losses in the optimal motion, respectively.

B. Internally dissipative friction

We substitute $\beta=1$ into Eq. (15) to get simply

$$w^* = E_i - E_f. \tag{21}$$

No energy is lost, all frictional losses are returned to the working fluid. Then the optimality condition, Eq. (13), results in

$$E_f^* = E_i \left(\frac{V_f}{V_i} \right)^{-R/C_\nu} \left[1 + \alpha(1-n) V_i \dot{V}_i^{m-1} \frac{(V_f/V_i)^f - 1}{f} \right]^{1/(1-n)} \tag{22}$$

and for sinusoidal motion

$$E_{f,\sin} = E_i \left(\frac{V_f}{V_i} \right)^{-R/C_\nu} \left[1 + \alpha(1-n) E_i^n \frac{V_i^m}{\tau^{m-1}} \pi^{m-1} \right.$$

$$\left. \times (r-1)^m r^{(n-1)R/C_\nu} B^2 F_1 \right]^{1/(1-n)}. \tag{23}$$

For small values of the frictional coefficient α we expand the last expression up to first order in α and define g_i as the ratio of the friction term of the sinusoidal motion relative to the optimal motion,

$$g_i = \frac{\pi^{m-1}(r-1)^m r^{-(n-1)R/C_\nu} B[(m+1)/2, (m+1)/2] {}^2F_1[(m+1)/2, (n-1)R/C_\nu, m+1; (r-1)/r]}{[(r^{1-d} - 1)/(1-d)]^{m-1}(r^f - 1)/f}. \tag{24}$$

For example, with $r=3$, $n=0.5$, and $m=2.0$, 2.5, and 3.0, the resulting values of g_i are 1.23, 1.40, and 1.63, respectively. At the same time the potential improvement of the maximal power output is 10% by applying the optimal motion compared to the conventional motion.

These relations with $\beta=1$ are valid for any value of $n \neq 1$. The case $n=1$ is special and must be treated separately, such that the following expression may be derived. Equation (13) leads to

$$E_f^* = E_i \left(\frac{V_f}{V_i}\right)^{-R/C_v} \exp[\alpha \dot{V}_i^m (r-1)] \qquad (25)$$

for the optimal motion with $d=0$. The sinusoidal motion yields

$$E_{f,\sin} = E_i \left(\frac{V_f}{V_i}\right)^{-R/C_v} \exp\left[\alpha \frac{V_i^m}{\tau^{m-1}} \pi^{m-1} \right.$$
$$\left. \times (r-1)^m B\left(\frac{m+1}{2}, \frac{m+1}{2}\right) \right]. \qquad (26)$$

For this case g_i is given by Eq. (20), the same equation for externally dissipative friction $\beta=0$ with $n=0$.

V. OPTIMAL MOTION FOR MINIMAL DISSIPATION

The optimal motions derived until now are for maximal power ($\delta=1$). The optimal motion for minimum frictional losses ($\delta=0$) is identical to the optimal motion for maximizing power only in the case of externally dissipative friction [$\beta=0$, Eq. (8)], and thus the equations derived in the previous section are valid for this case. For internal dissipation the optimal motion must be solved numerically for any n.

For the special case $n=0$ the solution is simple. Actually minimizing frictional losses leads to the constant speed optimal motion in the case of internally dissipative friction. The ratio between friction dissipation terms (sinusoidal/optimal) is then given by

$$g_i = \frac{\pi^{m-1}(r-1)(R/C_v+1)B[(m+1)/2,(m+1)/2]^2 F_1[(m+1)/2,(n-1)R/C_v,m+1;(r-1)/r]}{r - r^{-R/C_v}}. \qquad (27)$$

For illustration consider the values $n=0$, $r=3$, and $m=2.0$, 2.5, and 3.0. The resulting g_i are 1.24, 1.42, and 1.65, respectively.

Although the optimal motions are very different (see Fig. 2), the numerical values of g_i for maximizing power production are not much different from corresponding values of g_i for minimal losses.

VI. CONCLUSIONS

Optimal piston motion has been derived for adiabatic processes in the presence of friction. The type of friction and its effect on the optimal motion is studied. The dependence of viscosity on temperature is taken into account in the model such that the frictional dissipation depends on the internal energy to the power n and on the speed of the piston to the power m. The optimal motion is derived for two different objectives: maximizing power production and minimizing frictional losses. The frictional losses may be dissipated either externally or internally.

In the case of externally dissipative friction maximizing power is equivalent to minimizing frictional losses. The resulting optimal motion depends on the power n (temperature dependence of viscosity), and the optimal motion is no longer constant speed motion,[5] except in the case $n=0$ without temperature dependence. The optimal speed may increase or decrease with time depending on the sign of n.

In the case of internally dissipative friction, maximizing power leads to an optimal motion different from the optimal motion for minimizing frictional losses. In the latter case the solution has to be found numerically. For the special case $n=0$ the resulting optimal motion to minimize frictional losses is again constant speed.

Analytic expressions for calculating the ratio of the frictional dissipation term of the sinusoidal (conventional) motion relative to the optimal one are derived. This ratio depends on the compression ratio r, the powers n and m, and

R/C_v. For typical values $3 < r < 8$, $0 < n < 1$, $2 < m < 3$, and $R/C_v = 2/3$, this ratio is in the range of 1.24–1.65.

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