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Maximum power processes for multi-source endoreversible heat engines

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Abstract

The maximum power processes of multi-source endoreversible engines with stationary temperature reservoirs are investigated. We prove that the optimal solution is always time independent with a single hot and a cold engine contact temperature. The heat reservoirs fall into three groups: the hot reservoirs which are connected at all times for heat delivery, the cold reservoirs which are connected at all times for heat drain, and possibly a group of reservoirs at intermediate temperatures which are unused. This phenomenon is demonstrated for a three-source system. We find that for a commonly used class of heat transfer functions, including Newtonian, Fourier, and radiative heat transport, the efficiencies at maximum power are the same as for two-reservoir engines with appropriately chosen properties.

1. Introduction

Heat engines with several heat sources are common for many real-world applications such as industrial heat-recovery systems and solar energy installations. In such installations several different heat sources are present, which provide heat at different rates, and, even more importantly, at different temperatures. For instance, in solar energy installations these differences can come about because the angle towards the sun may differ or because some solar collectors might be at a larger distance from the central plant and thus the losses along the transport pipes cause a change in the effective temperature at the engine.

This paper investigates the performance limits for such engines. In particular, we study maximum power processes for an endoreversible engine [1–3] operating between several heat reservoirs. The average power output is used as a criterion of thermodynamic merit. Methods of averaged nonlinear programming [4–6], which are already well established for the optimization of thermal systems with two heat reservoirs [7–9], are applied to determine the maximum possible average power output and the corresponding optimal contact functions between the heat reservoirs and the power converting subsystem.

In this model, all irreversible processes are associated with the interactions between the heat engine and the heat reservoirs while the processes inside the reservoirs and power converting subsystem are reversible (i.e. the endoreversibility hypothesis [1]). This study thus complements earlier work on staged and combined systems (see for instance, [10–19]).

2. Model

A model of a multi-source endoreversible heat engine is depicted in figure 1. It consists of a power converting subsystem operating reversibly between two temperature

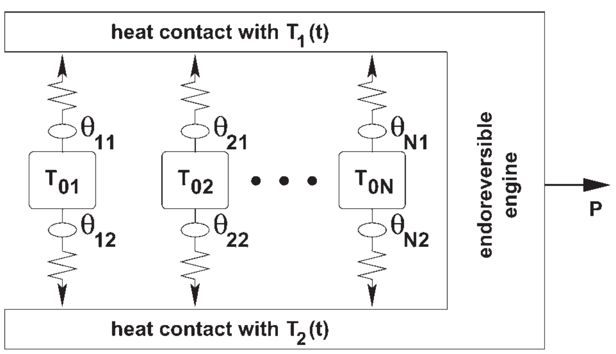

Figure 1. A multi-source endoreversible engine.

contacts $T_1(t)$ and $T_2(t)$ and of N heat sources/sinks at constant temperatures T_{0i} where $i \in [1, N]$. The heat transfer laws between source i and engine contact $\alpha \in \{1, 2\}$ have the form

$$\tilde{q}_{i\alpha}(T_{0i}, T_\alpha, \theta_{i\alpha}) = \theta_{i\alpha} q_{i\alpha}(T_{0i}, T_\alpha), \qquad i \in [1, N]. \tag{1}$$

The contact functions $\theta_{i\alpha}$ describe the extent of the contact between the reservoir and the engine. They are equal to one if the ith reservoir is fully connected to contact α and are equal to zero if there is no contact. Thus, $0 \leqslant \theta_{i\alpha}(t) \leqslant 1$.

In the following, we assume that the functions $q_{i\alpha}(T_{0i}, T_\alpha)$ show the standard behaviour of heat transfer as a function of T_{0i} and T_α, i.e. heat flows from high to low temperature: $q_{i\alpha}(T_{0i}, T_\alpha) > 0$ if $T_{0i} > T_\alpha$, $q_{i\alpha}(T_{0i}, T_\alpha) < 0$ if $T_{0i} < T_\alpha$, and $q_{i\alpha}(T_{0i}, T_\alpha) = 0$ if $T_{0i} = T_\alpha$.

3. Power optimization

Our aim is to determine maximum power processes for this multi-source engine as well as the efficiency at maximum power. The system is operated under cyclic conditions with a fixed cycle time τ_{tot}, i.e. internal energy and entropy of the endoreversible engine is the same at the beginning and the end of the cycle. The system is optimized for maximum average power output $\bar{P}$. This is equivalent to the maximization of the time-averaged sum of the heat flows

$$q_\Sigma(\boldsymbol{T}_0, T_1(t), T_2(t), \boldsymbol{\theta}_1(t), \boldsymbol{\theta}_2(t)) = \sum_{i=1}^{N} \tilde{q}_{i1}(T_{0i}, T_1, \theta_{i1}) + \tilde{q}_{i2}(T_{0i}, T_2, \theta_{i2}), \tag{2}$$

where the vectors $\boldsymbol{\theta}_\alpha$ of the contact functions and the vector $\boldsymbol{T}_0$ of the reservoir temperatures are defined as

$$\boldsymbol{\theta}_\alpha = (\theta_{1\alpha}, \theta_{2\alpha}, \ldots, \theta_{N\alpha}), \tag{3}$$

$$\boldsymbol{T}_0 = (T_{01}, T_{02}, \ldots, T_{0N}). \tag{4}$$

The optimization problem can be stated formally as the maximization of the functional

$$I = \bar{P} = \overline{q_\Sigma}\,[\boldsymbol{T}_0, T_1(t), T_2(t), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2], \tag{5}$$

$$I = \frac{1}{\tau_{\mathrm{tot}}} \int_0^{\tau_{\mathrm{tot}}} \sum_{i=1}^{N} (\tilde{q}_{i1}(T_{0i}, T_1, \theta_{i1}) + \tilde{q}_{i2}(T_{0i}, T_2, \theta_{i2}))\, \mathrm{d}t \to \max_{\{T_\alpha, \boldsymbol{\theta}_\alpha\}} \tag{6}$$

by varying the engine contact temperatures T_1 and T_2 and all the contact functions $\boldsymbol{\theta}_\alpha$ subject to the restriction of entropy balance

$$\overline{s_\Sigma} = \overline{s_\Sigma}\,[\boldsymbol{T}_0, T_1(t), T_2(t), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2], \tag{7}$$

$$\overline{s_\Sigma} = \frac{1}{\tau_{\mathrm{tot}}} \int_0^{\tau_{\mathrm{tot}}} \sum_{i=1}^{N} \left(\frac{\tilde{q}_{i1}(T_{0i}, T_1, \theta_{i1})}{T_1} + \frac{\tilde{q}_{i2}(T_{0i}, T_2, \theta_{i2})}{T_2} \right) \mathrm{d}t = 0. \tag{8}$$

This restriction is due to the cyclic operation of the system.

The temperatures $0 < T_1(t), T_2(t) < \infty$ of the endoreversible engine and the elements of the vector $\boldsymbol{\theta}_\alpha$ of contact functions are the $2N + 2$ controls of the system. These elements satisfy the conditions

$$0 \leqslant \theta_{i\alpha}(t) \leqslant 1; \qquad i \in [1, N], \quad \alpha \in \{1, 2\}. \tag{9}$$

Equations (6)–(9) can be optimized either by using optimal control theory or by a special kind of nonlinear programming. As the objective as well as the constraint are both in the form of a time average over a given interval, we will here use averaged nonlinear programming. The Lagrangian for this problem has the form [4–6]

$$\mathcal{L} = \sum_{i=1}^{N} \left(\tilde{q}_{i1}(T_{0i}, T_1, \theta_{i1}) \left[1 - \frac{\lambda}{T_1}\right] + \tilde{q}_{i2}(T_{0i}, T_2, \theta_{i2}) \left[1 - \frac{\lambda}{T_2}\right] \right) \to \max_{\{T_\alpha, \boldsymbol{\theta}_\alpha\}}, \tag{10}$$

where λ is a kind of Lagrange multiplier associated with the entropy balance, equation (8).

The theory of averaged programming [4, 5] states that the solution for the optimal controls are piecewise constant functions taking values out of a set of no more than $m + 1$ *base points* where m is the number of averaged constraints of the problem (see [6], p 78 ff).

The first point to note is thus that T_1 and T_2 are piecewise constant in time. The second point is that there is only one averaged constraint (8) here, and consequently there are no more than two base points for the controls and in particular for the temperatures $T_1(t)$ and $T_2(t)$. Let us call them T_{h} and T_{c}. As the optimization problem set up above is fully symmetric in the two contacts 1 and 2, the two base points are either of the form $(T_1, T_2) = (T_{\mathrm{h}}, T_{\mathrm{c}})$ and $(T_1, T_2) = (T_{\mathrm{c}}, T_{\mathrm{h}})$, or there is only one base point which then must have $T_1 = T_2$. In the latter case the engine does not produce any power, so we exclude this case from our further considerations. In the first case we need to discuss only one of the base points and can thus assume $T_{\mathrm{c}} < T_{\mathrm{h}}$ without loss of generality.

From a physical point of view this means that, even though in principle the temperatures at the contacts of the endoreversible engine could change, it is optimal to run the engine in a stationary mode with fixed temperatures.

4. Optimal contact functions

We now turn to the variation of $\mathcal{L}$ with respect to the controls $\theta_{i\alpha}$. The Lagrangean $\mathcal{L}$ depends linearly on each $\theta_{i\alpha}$, so that $\mathcal{L}$ will attain its maximum only at the boundary values {0,1} of the admissible range of $\theta_{i\alpha}$. This determines a rule for the contact functions:

$$\theta_{i\alpha}(T_{0i}, T_{i\alpha}) = \begin{cases} 1, & \text{if } \left[1 - \dfrac{\lambda}{T_\alpha}\right] q_{i\alpha}(T_{0i}, T_\alpha) > 0, \\[2ex] 0, & \text{if } \left[1 - \dfrac{\lambda}{T_\alpha}\right] q_{i\alpha}(T_{0i}, T_\alpha) < 0, \end{cases} \qquad i \in [1, N], \quad \alpha \in \{1, 2\}. \tag{11}$$

Let us take a closer look at this rule. $q_{i\alpha}(T_{0i}, T_\alpha) < 0$ means that the contact α connects with reservoirs which serve as heat sinks and thus fulfil the condition $T_{0i} < T_\alpha$. In the opposite case, i.e. $q_{i\alpha}(T_{0i}, T_\alpha) > 0$, this implies the condition $T_{0i} > T_\alpha$. The working fluid then connects to reservoirs which act as heat sources.

We now distinguish three cases:

$T_h > T_c > \lambda$:

$$\left[1-\frac{\lambda}{T_h}\right] > 0 \Rightarrow \begin{cases} \theta_{ih}(T_{0i}, T_h, \lambda) = 1, \\ \quad \text{if } q_{ih} > 0, \text{ i.e. } T_{0i} > T_h, \\ \theta_{ih}(T_{0i}, T_h, \lambda) = 0, \\ \quad \text{if } q_{ih} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_h, \end{cases} \tag{12}$$

$$\left[1-\frac{\lambda}{T_c}\right] > 0 \Rightarrow \begin{cases} \theta_{ic}(T_{0i}, T_c, \lambda) = 1, \\ \quad \text{if } q_{ic} > 0, \text{ i.e. } T_{0i} > T_c, \\ \theta_{ic}(T_{0i}, T_c, \lambda) = 0, \\ \quad \text{if } q_{ic} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_c. \end{cases} \tag{13}$$

In this case all $\tilde{q}_{i\alpha}$ are either positive or vanishing, which means that due to the entropy constraint all $\tilde{q}_{i\alpha}$ have to equal zero and thus no power is produced. We therefore exclude this case from our further consideration.

$\lambda > T_h > T_c$:

$$\left[1-\frac{\lambda}{T_h}\right] < 0 \Rightarrow \begin{cases} \theta_{ih}(T_{0i}, T_h, \lambda) = 0, \\ \quad \text{if } q_{ih} > 0, \text{ i.e. } T_{0i} > T_h, \\ \theta_{ih}(T_{0i}, T_h, \lambda) = 1, \\ \quad \text{if } q_{ih} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_h, \end{cases} \tag{14}$$

$$\left[1-\frac{\lambda}{T_c}\right] < 0 \Rightarrow \begin{cases} \theta_{ic}(T_{0i}, T_c, \lambda) = 0, \\ \quad \text{if } q_{ic} > 0, \text{ i.e. } T_{0i} > T_c, \\ \theta_{ic}(T_{0i}, T_c, \lambda) = 1, \\ \quad \text{if } q_{ic} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_c. \end{cases} \tag{15}$$

In this case all $\tilde{q}_{i\alpha}$ are either negative or vanishing, which means that due to the entropy constraint all $\tilde{q}_{i\alpha}$ have to equal zero and thus no power is produced. Again we, therefore, exclude this case from our further consideration.

$T_h > \lambda > T_c$:

$$\left[1-\frac{\lambda}{T_h}\right] > 0 \Rightarrow \begin{cases} \theta_{ih}(T_{0i}, T_h, \lambda) = 1, \\ \quad \text{if } q_{ih} > 0, \text{ i.e. } T_{0i} > T_h, \\ \theta_{ih}(T_{0i}, T_h, \lambda) = 0, \\ \quad \text{if } q_{ih} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_h, \end{cases} \tag{16}$$

$$\left[1-\frac{\lambda}{T_c}\right] < 0 \Rightarrow \begin{cases} \theta_{ic}(T_{0i}, T_c, \lambda) = 0, \\ \quad \text{if } q_{ic} > 0, \text{ i.e. } T_{0i} > T_c, \\ \theta_{ic}(T_{0i}, T_c, \lambda) = 1, \\ \quad \text{if } q_{ic} \leqslant 0, \text{ i.e. } T_{0i} \leqslant T_c. \end{cases} \tag{17}$$

In this case only reservoirs with positive $q_{ih}(T_{0i}, T_h)$ are connected to the engine contact at T_h, i.e. those with $T_{0i} > T_h$. In the same way only negative $q_{ic}(T_{0i}, T_c)$ are connected to the engine contact at T_c, i.e those with $T_{0i} < T_c$.

Note that due to the stationarity of T_h, T_c, and λ, the contact functions are also independent of time.

5. Hot, cold, and unused reservoirs

We can now draw a number of interesting and surprising conclusions from the structure of the optimal contact functions. First of all it is clear that a reservoir will be at most connected to one heat contact. Those reservoirs with temperatures above the hot contact temperature will deliver heat, those with temperatures below the the cold temperature contact will receive heat. All reservoirs with temperatures in the range between T_c and T_h are therefore never connected during a cycle; these reservoirs are referred to as *unused reservoirs*.

As a consequence the set of N heat reservoirs is divided into three subsets R_h, R_c, and R_u of *hot*, *cold*, and *unused* reservoirs, respectively. Note that the unused set may be empty depending on the temperatures of the reservoir. Also, it is clear that the hottest and coldest reservoirs are always active in a finite power producing solution.

6. Construction of heating and cooling functions

To determine the heat flows resulting from the above rule, the heat transfer functions for each reservoir are separated into heat input and output functions,

$$q_{ih}^{+}(T_{0i}, T_h) = \begin{cases} q_{ih}(T_{0i}, T_h), & \text{if } T_{0i} \geqslant T_h, \\ 0, & \text{if } T_{0i} < T_h, \end{cases} \tag{18}$$

$$q_{ic}^{-}(T_{0i}, T_c) = \begin{cases} 0, & \text{if } T_{0i} > T_c, \\ q_{ic}(T_{0i}, T_c), & \text{if } T_{0i} \leqslant T_c, \end{cases} \tag{19}$$

for each $i \in [1, N]$. The total rate of heat input to and from the endoreversible engine are calculated as the sum of all contributions $q_{ih}^{+}(T_{0i}, T_h)$ and $q_{ic}^{-}(T_{0i}, T_c)$,

$$q^{+}(\boldsymbol{T}_0, T_h) = \sum_{i=1}^{N} q_{ih}^{+}(T_{0i}, T_h), \tag{20}$$

$$q^{-}(\boldsymbol{T}_0, T_c) = \sum_{i=1}^{N} q_{ic}^{-}(T_{0i}, T_c). \tag{21}$$

The heat exchange causes an entropy change of the working fluid. The rates of entropy flow to the working fluid are easily obtained by dividing the corresponding heat exchange rate by the current temperature of the working fluid. Specifically, the rates of entropy change are

$$\begin{aligned} s^{+}(\boldsymbol{T}_0, T_h) &= \frac{q^{+}(\boldsymbol{T}_0, T_h)}{T_h}, \\ s^{-}(\boldsymbol{T}_0, T_c) &= \frac{q^{-}(\boldsymbol{T}_0, T_c)}{T_c}. \end{aligned} \tag{22}$$

7. Optimal temperatures for the engine contacts

The Lagrange function (10) can now be expressed in terms of the above-defined functions:

$$\mathcal{L} = q^{+}(\boldsymbol{T}_0, T_h)\left[1-\frac{\lambda}{T_h}\right] + q^{-}(\boldsymbol{T}_0, T_c)\left[1-\frac{\lambda}{T_c}\right]. \tag{23}$$

Already at this point it is apparent that the optimization of a multi-source heat engine is equivalent to the optimization of a two-reservoir heat engine with the heat transfer functions defined above.

In order to determine the optimal temperatures we use the optimality condition $\partial\mathcal{L}/\partial T_h = 0$ and $\partial\mathcal{L}/\partial T_c = 0$, which can be rewritten as

$$\lambda = \frac{\partial q^{+}(\boldsymbol{T}_0, T_h)}{\partial T_h} T_h \left(\frac{\partial q^{+}(\boldsymbol{T}_0, T_h)}{\partial T_h} - \frac{q^{+}(\boldsymbol{T}_0, T_h)}{T_h}\right)^{-1} \tag{24}$$

and

$$\lambda = \frac{\partial q^{-}(\boldsymbol{T}_0, T_c)}{\partial T_c} T_c \left(\frac{\partial q^{-}(\boldsymbol{T}_0, T_c)}{\partial T_c} - \frac{q^{-}(\boldsymbol{T}_0, T_c)}{T_c}\right)^{-1}. \tag{25}$$

Together with the constraint on the entropy as given by equation (8), these equations determine the values of λ, T_h, T_c for given laws of heat conduction $q_i(T_{0i}, T)$ and reservoir temperatures T_{0i}. These equations are used to analytically or numerically calculate optimal solutions.

8. A special class of heat transport equations

We now restrict our analysis to an important subclass of heat transport equations. This class is characterized by a special structure in which the net heat flow is a sum or difference of two terms, where each system contributes to the heat flow based on just its own temperature:

$$q_{i\alpha}(T_{0i}, T_\alpha) = q_{i\alpha}^{(r)}(T_{0i}) + q_{i\alpha}^{(s)}(T_\alpha). \tag{26}$$

This structure includes, for instance, Newtonian heat transfer, Fourier heat transfer, and heat transport by radiation. An example of a transport rule that does not comply with this structure is due to Anand:

$$q_{i\alpha}(T_{0i}, T_\alpha) = (T_{0i} - T_\alpha)^\gamma. \tag{27}$$

For transport equations of the form (26) the summed up heat flows obey

$$q^+(\boldsymbol{T}_0, T_h) = \sum_{i\in R_h} q_{ih}^{(r)}(T_{0i}) + \sum_{i\in R_h} q_{ih}^{(s)}(T_h) = q_h^{(r)}(\boldsymbol{T}_0) + q_h^{(s)}(T_h), \tag{28}$$

$$q^-(\boldsymbol{T}_0, T_c) = \sum_{i\in R_c} q_{ic}^{(r)}(T_{0i}) + \sum_{i\in R_c} q_{ic}^{(s)}(T_c) = q_c^{(r)}(\boldsymbol{T}_0) + q_c^{(s)}(T_c). \tag{29}$$

Thus, the system behaves as if connected to only two reservoirs.

If further the multiple sources are coupled to the engine by transport rules with the same T dependent function f, i.e.

$$q_{i\alpha}(T_{0i}, T_\alpha) = a_{i\alpha}(f(T_{0i}) - f(T_\alpha)), \tag{30}$$

then the summed up heat transfer functions q^+ and q^- have the same structure

$$q^+(\boldsymbol{T}_0, T_h) = a_h(f(T_{0h}) - f(T_h)), \tag{31}$$

$$q^-(\boldsymbol{T}_0, T_c) = a_c(f(T_{0c}) - f(T_c)) \tag{32}$$

with $a_\alpha = \sum_{i\in R_\alpha} a_{i\alpha}$ and

$$T_{0\alpha} = f^{-1}\left(\frac{\sum_{i\in R_\alpha} a_{i\alpha} f(T_{0i})}{\sum_{i\in R_\alpha} a_{i\alpha}}\right). \tag{33}$$

Note that Newtonian, Fourier, and radiation heat transfer have this property.

As an example, let us consider a system with three heat reservoirs. The temperatures of these reservoirs are T_{01}, T_{02}, and T_{03}. Whenever they are connected to either of the two contacts of the engine, the heat exchange is assumed to obey a linear transport law,

$$q_i(T_{0i}, T) = \alpha_i(T_{0i} - T). \tag{34}$$

The temperatures of the three reservoirs are chosen to be $T_{01} = 1$, $T_{02} = 1.6$, and $T_{03} = 4$. We set the heat conductances $\alpha_1 = \alpha_3 = 1$ by an appropriate choice of units. We then vary the heat conductance α_2 of the intermediate reservoir T_{02} between 0 and 5 in steps of 0.1 and study how the behaviour of the system changes.

The power output of these systems have been numerically determined. In this particularly simple example we used a less general method than the one introduced above in equations (24) and (25). Here we expressed the heat flows at the two contacts in terms of the respective entropy flows: $q^+(s^+)$ and $q^-(s^-)$. Whether such an approach is possible depends on the heat transport equations: one cannot always determine the heat flow as a single-valued function of the entropy flow. If one can, then the advantage of this approach lies in the fact that the entropy constraint can explicitly be fulfilled by setting $s^+ = -s^-$. In our case the power output $q^+(s^+) + q^-(-s^+)$ was easily optimized by a line search varying s^+. However, the reader should be aware that the power output might not be a unimodal function.

In the following we analyse the dependence of the results on the value of the heat conductance α_2. The system shows a cross-over behaviour at a critical heat conductance. This cross-over is intimately connected to the phenomenon of unused heat reservoirs.

In figure 2, the indicator function ind(α_2) shows how the intermediate heat reservoir is used. For small heat conductances the second reservoir is not used at all, here indicated by ind $= 0$, only reservoir 1 is connected to the cold contact and reservoir 3 is connected to the hot contact. With increasing heat conductance α_2 the reservoir becomes more and more important and—at the critical point—it is connected to the cold engine contact together with reservoir 1, here indicated by ind $= 1$.

Figure 3 shows the resulting maximal power output as a function of α_2. The dots represent the results of the numerical optimization. With increasing α_2 the power stays constant at 0.5 until reservoir 2 is switched on, then the power increases.

In figure 4 the corresponding efficiency at maximum power is shown. The behaviour discussed above can be understood in terms of a Curzon–Ahlborn engine operating between two heat baths.

Initially the system uses only two of the three heat reservoirs: the hottest and the coldest. Then power and efficiency at maximum power are those of the Curzon–Ahlborn engine:

$$P_{opt} = \frac{\alpha_1\alpha_3}{\alpha_1 + \alpha_3}(\sqrt{T_{01}} - \sqrt{T_{03}})^2, \tag{35}$$

$$\eta_{CA} = 1 - \sqrt{\frac{T_{01}}{T_{03}}} = 1 - \sqrt{\frac{1}{4}} = 0.5. \tag{36}$$

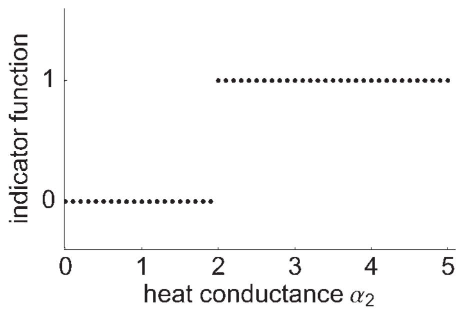

Figure 2. The indicator function ind(α_2) takes the value equal to the reservoir together with which reservoir 2 is used or zero in case it is not used at all.

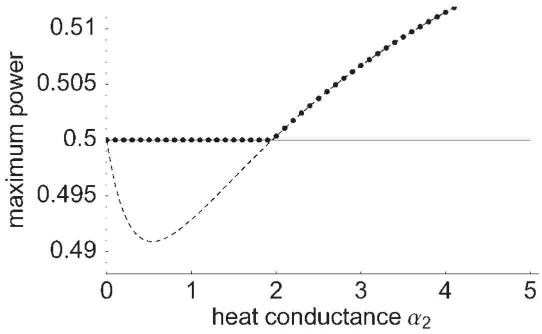

Figure 3. The maximal power output as a function of heat conductance α_2. Dots show the numerical solution for three reservoirs at $T_{01} = 1$, $T_{02} = 1.6$, and $T_{03} = 4$ and conductances $\alpha_1 = \alpha_3 = 1$. The solid line is the optimal solution of a Curzon–Ahlborn engine operating between reservoirs 3 and 1 only, while the dashed line is the optimal solution for operation between reservoirs 3 and the combined reservoirs 1 and 2.

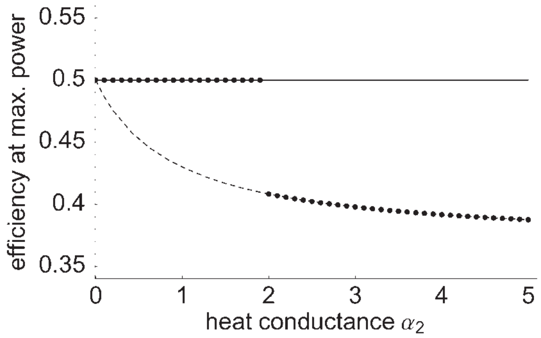

Figure 4. The efficiency at maximum power. Same nomenclature as in figure 3.

These values are displayed as a solid line in figures (3) and (4). The dashed line corresponds to a Curzon–Ahlborn engine which operates between reservoir 3 as the high temperature reservoir and reservoirs 2 and 1 combined as the cold reservoir.

The results demonstrate very nicely that as soon as the dashed line crosses the solid line, reservoir 2 is put into use. This reservoir shows a cross-over from unused to used at which the power output is non-differentiable but continuous, not unlike a second order phase transition. The efficiency, on the other hand changes, in a step-like fashion.

9. Summary

In this paper, a power-producing endoreversible engine which exchanges heat with several (two or more) constant temperature heat sources was considered. The heat transport is general and includes heat transfer processes obeying Newtonian, Fourier, and radiative heat transport. One of the interesting questions for such systems is how the different heat reservoirs are used in an optimal fashion. The system was optimized for maximum power output in case of cyclic operation using averaged nonlinear programming methods. The optimal solution shows a number of interesting properties: first, it was shown that there are conditions where some reservoirs should not be connected at all in order to achieve an optimal performance of the system. These reservoirs are referred to as *unused reservoirs*. The hottest and the coldest reservoir will always be used.

The second important finding is that the optimal operation is a stationary one, where the *used* reservoirs and the power-producing subsystem contact temperatures are time-independent.

We analysed a special class of heat transfer functions which includes the often used Newtonian, Fourier, and radiative heat transport. For this class of transport laws we showed that a multi-source endoreversible engine is equivalent to a two-source engine with a particular choice of transport laws.

These results were finally exemplified by a simple three-heat source engine for which the heat conductance to the reservoir with the intermediate temperature was varied.

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