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Optimal Processes for Controllable Oscillators

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Abstract—We consider the problem of optimal parametric control for a single oscillator or an ensemble of oscillators due to a change in one of the coefficients of the system of equations characterizing them. We obtain solutions for the problem of finding the maximal change in the energy of oscillations for a given time.

Keywords: parametric control, single oscillator, oscillator ensemble, speed dependence

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1. INTRODUCTION

The problem of controlling the oscillations of a pendulum for the case when the control action is a force that influences the acceleration or the speed of the pendulum additively differs little from the motion control problem studied already by A.A. Feldbaum. Meanwhile, one can often observe how the swings are accelerating or braking by changing the distance from the point of suspension to the center of gravity of the pendulum. Formally, this corresponds to parametric control, when the control action is one of the coefficients in the differential equation. Such a problem has been considered in [1] with regard to the linearized pendulum. That work obtained a simple but inexact solution. Below we present a solution to the problem of parametric control of a single oscillator and shows the fundamental difference that arises when solving a similar problem for an ensemble of oscillators.

An ensemble of oscillators is a system of synchronized linearized pendulums that have the same frequency but different phases of oscillations. The oscillation energy of each oscillator in the ensemble is mechanical, and in this respect it is similar to a single oscillator, but the oscillation phases of each oscillator are different and the system can be controlled only at the macro level, by affecting the environmental parameters that determine the common oscillation frequency for the ensemble. In this respect, an ensemble of oscillators is similar to a macrosystem, and one of the variables characterizing it is von Neumann entropy (see [2]).

One physical system adequate to the considered model is a crystalline body. The control is laser radiation [2]. The change in the internal energy of the system corresponds to its heating or cooling. Moreover, unlike such classical macrosystems as an ideal gas, an ensemble of oscillators can be heated or cooled in finite time without changing entropy (adiabatically). The works [3, 4] show that the duration of adiabatic cooling is limited from below and find this lower bound. Below we show that it is possible to reduce the energy of oscillations of an ensemble adiabatically only up to a certain limit, even over an arbitrarily long time. In the same problem for a single oscillator, the oscillation energy can be made arbitrarily small. In [5], the performance problem for an ensemble of oscillators is considered as an illustration of the capabilities of the method of substitution of phase variables.

2. A SINGLE OSCILLATOR

Let p be the speed and q be the displacement of an ideal linearized pendulum. Its motion is characterized by equations

$$\dot{q} = p, \quad \dot{p} = -uq, \quad p(0) = p_0, \quad q(0) = q_0, \tag{2.1}$$

where $u \geqslant 0$ is a parameter depending on the mass of the pendulum and the length of its suspension (the distance from the point of suspension to the center of gravity). Physically, it is equal to the square of the frequency of its natural oscillations. The suspension length can be changed by changing the position of the center of gravity. Thus, $u(t)$ is the control bounded by the condition

$$0 < u_1 \leqslant u \leqslant u_2. \tag{2.2}$$

The parametric control problem for an oscillator has been considered in [1, 5] as a problem where one wants to maximize performance.

Below we show that the problem of minimizing transition time to a given energy level and minimizing (maximizing) the energy increment over a given time interval are not equivalent, since the dependence of the maximum achievable energy on the process duration is not a strictly monotonic function.

The total energy of the oscillator is

$$E(t) = p^2 + uq^2. \tag{2.3}$$

For any constant value u, the rate of change of E is equal to zero by virtue of Eqs. (2.1), i.e., $E(t) = \text{const}$.

At the initial moment of time

$$E_0 = p_0^2 + u_0 q_0^2.$$

At the final moment τ,

$$\overline{E} = E(\tau) = p^2(\tau) + u(\tau)q^2(\tau) = \overline{p}^2 + u(\tau)\overline{q}^2. \tag{2.4}$$

Since $u(\tau)$ can be changed abruptly, depending on the formulation the optimality criterion in the problem of the maximum energy change will take the form

$$\overline{E} = \overline{p}^2 + u_2\overline{q}^2 \longrightarrow \max \tag{2.5}$$

or

$$\overline{E} = \overline{p}^2 + u_1\overline{q}^2 \longrightarrow \min. \tag{2.6}$$

To be definite, in what follows we consider the problem with criterion (2.6), and without loss of generality we let $u_1 = 1$, so $1 \leqslant u \leqslant u_2/u_1$. Energy $\overline{E}$ can be rewritten in integral form as

$$\overline{E} = E_0 + \int_0^\tau \frac{d}{dt}(p^2 + q^2)dt = E_0 - 2\int_0^\tau pq(u-1)dt \longrightarrow \min. \tag{2.7}$$

From this expression, the work [1] makes an erroneous conclusion that for even (2 and 4) quadrants of the plane p, q, when the product of these variables is negative, it is necessary that $u^* = 1$, and for odd quadrants (1 and 3), where $pq > 0$, one should choose $u^* = \frac{u_2}{u_1}$. Relations (2.1) between control and state variables were not taken into account.

Before solving the problem (2.7), (2.1), we simplify it by passing to new variables. Since the right-hand part of Eq. (2.1) can change sign, we introduce a new variable $z(p, q)$ so that its speed along the system trajectories does not change sign.

Phase trajectories of the oscillator for any admissible values of u rotate counterclockwise, so we select as z the expression

$$z = \arctan\left(-\frac{p}{q}\right), \quad z_0 = \arctan\left(-\frac{p_0}{q_0}\right). \tag{2.8}$$

The rate of change of this variable is greater than or equal to one:

$$\dot{z} = \frac{1}{1 + \frac{p^2}{q^2}} \frac{uq^2 + p^2}{q^2} = \frac{u + \tan^2 z}{1 + \tan^2 z}. \tag{2.9}$$

We next replace the energy E with a variable monotonically associated with it

$$e = \ln(p^2 + q^2),$$

so that

$$\dot{e} = 2\frac{pq(1-u)}{p^2+q^2} = 2\frac{\tan z(u-1)}{1+\tan^2 z}, \quad e_0 = \ln E_0. \tag{2.10}$$

Note that the right-hand side of Eq. (2.9) is positive, and the variable e does not appear in the right-hand side of (2.9), (2.10), i.e., Eq. (2.10) in the terminology of [6] is Lyapunov and can be rewritten in integral form, as it is done below.

We substitute

$$dt = \frac{(1+\tan^2 z)dz}{\tan^2 z + u}$$

and rewrite the problem of braking the oscillator in the form

$$e(\tau) = e(\overline{z}) = e_0 + 2\int_{z_0}^{\overline{z}} \frac{\tan z(u-1)dz}{u + \tan^2 z} \longrightarrow \min_{u(z),\overline{z}} \tag{2.11}$$

under constraint

$$\int_{z_0}^{\overline{z}} \frac{(1+\tan^2 z)dz}{\tan^2 z + u} = \tau. \tag{2.12}$$

The solution for this problem is a dependence $u^*(z)$ and a value $\overline{z}^*$. Note that $z > 0$ for even and $z < 0$ for odd quadrants of the plane p, q, therefore, if we ignore condition (2.12) the solution from [1] is correct. It is closer to the correct solution of the problem when τ is larger, i.e., constraint (2.12) is less significant.

For the resulting problem (2.11), (2.12) with one integral condition there exists a nonzero vector λ such that the Lagrange function L on the optimal solution is minimal for u. For a non-degenerate solution ($\lambda_0 \neq 0$) L takes the form

$$L = \frac{\tan z(u-1)}{(u+\tan^2 z)} + \lambda\frac{1+\tan^2 z}{u+\tan^2 z}. \tag{2.13}$$

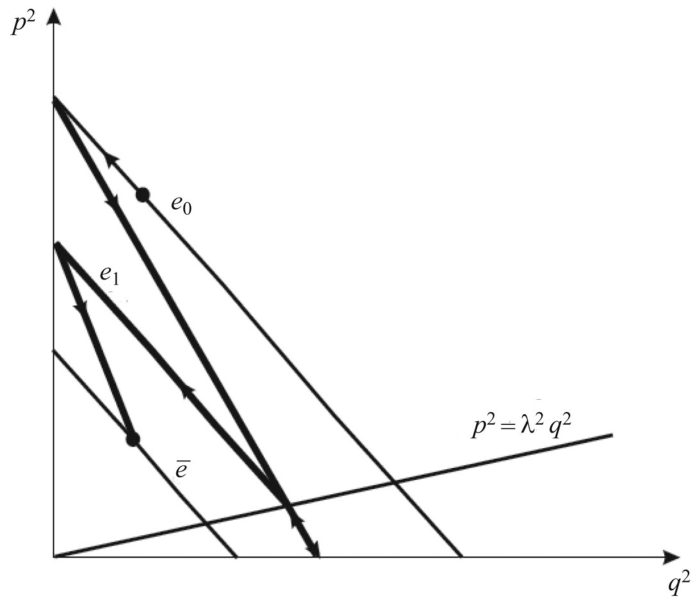

Fig. 1. Change of the oscillator energy along the optimal trajectory on the plane p^2, q^2.

The derivative of L with respect to u is

$$\frac{\partial L}{\partial u} = \frac{1 + \tan^2 z}{(u + \tan^2 z)^2}(\tan z - \lambda),$$

and its sign coincides with the sign of the second factor. Optimal control is maximal ($u^* = \frac{u_2}{u_1}$) when $(\tan z - \lambda) < 0$. For $(\tan z - \lambda) > 0$ it is minimal ($u^* = 1$).

The factor λ is equal to $-\frac{de^*(\tau)}{d\tau}$ (see [7]), where $e^*(\tau)$ is the minimum energy value of the oscillations that can be achieved in time τ. Since the value of e^* decreases with an increase in the admissible duration, we have $\lambda > 0$. The switching line is a straight line with slope $-\lambda$. It expands the sectors where control is maximized: these are not only the first and third quadrants, where $\tan z < 0$, but also parts of the second and fourth where $\tan z < \lambda$.

The second switching line is the Y-axis, on which $\tan z$ has a discontinuity.

For $z = \overline{z}$, the control u^* abruptly becomes equal to one.

It is convenient to depict the optimal process on a plane with coordinates p^2 and q^2, since the direct control values on this plane correspond to straight lines with slopes -1 and $-u_2/u_1$. If the initial state on the p, q plane lies to the right of the Y-axis and above the switching line $p = -\lambda q$ or to the left of this axis and below the switching line, then the optimal control at time $t = 0$ takes value $u_{\max} = u_2/u_1$, the trajectory intersects the X-axis ($p = 0$), and the system moves to the switching line $p^2 = \lambda^2 q^2$ (see Fig. 1), after which the control takes the value $u^* = 1$ and remains unchanged until the control switches on the Y-axis ($q = 0$).

If the initial conditions are such that p_0, q_0 are to the left of the Y-axis and above the switching line or to the right of the Y-axis and below this line, then $u^* = 1$ and the system first moves towards the Y-axis, and then it starts moving with a slope $-u_{\max} = -u_2/u_1$ up to the switching line. In addition, in each ith cycle that begins and ends on the Y-axis, the degree of attenuation

$$\varphi = \frac{e_{i-1}}{e_i} = \frac{\lambda^2 + u_{\max}}{\lambda^2 + 1} > 1 \tag{2.14}$$

does not depend on i.

The duration of such a cycle $\Delta\tau_i$ also depends on λ and does not depend on i. In the absence of restrictions on the duration of the process, the energy can be made arbitrarily small.

$$\Delta = \Delta\tau_i = \frac{\pi}{2}\left(1+\frac{1}{\sqrt{u_{\max}}}\right)+\frac{1}{\sqrt{u_{\max}}}\arctan\frac{\lambda}{\sqrt{u_{\max}}}-\arctan\lambda. \tag{2.15}$$

The ratio φ/Δ is maximized by the value of λ satisfying

$$\arctan\lambda-\frac{1}{\sqrt{u_{\max}}}\arctan\frac{\lambda}{\sqrt{u_{\max}}}=\frac{\pi}{2}\left(1+\frac{1}{\sqrt{u_{\max}}}\right)-\frac{1}{2\lambda}. \tag{2.16}$$

The left-hand side of this equation is negative, and the right-hand side is positive with $\lambda < 0$, so it has only a positive root. Since the left-hand side of the equation grows with λ and tends to the value $\frac{\pi}{2}\left(1-\frac{1}{\sqrt{u_{\max}}}\right)$, and the right-hand side increases monotonously up to a larger value $\frac{\pi}{2}\left(1+\frac{1}{\sqrt{u_{\max}}}\right)$, the solution is unique.

3. ENSEMBLE OF OSCILLATORS

Consider the problem of minimizing the oscillation energy in a given time τ for an ensemble of oscillators.

Suppose that the ith oscillator is characterized by Eqs. (2.1) that relate its displacement q_i and impulse p_i with each other and with the common frequency of the ensemble ω,

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\omega^2 q_i, \quad i = 1,\ldots,N. \tag{3.1}$$

With a choice of units we can set the oscillator mass equal to $m = 1$.

Macro-variables characterizing the ensemble [3]:

—energy

$$E(t) = \frac{1}{2}\sum_{i=1}^{N}(p_i^2+\omega^2 q_i^2), \tag{3.2}$$

—Lagrangian

$$L(t) = \frac{1}{2}\sum_{i=1}^{N}(p_i^2-\omega^2 q_i^2), \tag{3.3}$$

—correlation

$$C(t) = \omega(t)\sum_{i=1}^{N}p_i q_i. \tag{3.4}$$

Denoting $\dot{\omega}/\omega$ by u, we can write a system characterizing the dynamics of macro variables [2] as

$$\begin{gathered}\dot{E} = u(E-L), \quad E(0) = E_0,\\ \dot{L} = -u(E-L)-2\omega C, \quad L(0) = L_0,\\ \dot{C} = 2\omega L - uC, \quad C(0) = C_0,\\ \dot{\omega} = u\omega, \quad \omega_1 \leqslant \omega \leqslant \omega_2, \quad \omega(0) = \omega_0.\end{gathered} \tag{3.5}$$

Since by virtue of Eqs. (3.1)

$$\frac{d}{dt}(p_i q_j - p_j q_i) = -\omega^2 q_i q_j + p_i p_j + \omega^2 q_j p_i - p_j p_i = 0 \quad \forall i, j,$$

there must be a relationship between state variables in (3.5). Indeed, it is easy to see that

$$X = \frac{E^2 - L^2 - C^2}{\omega^2} = X_0 = \frac{E_0^2 - L_0^2 - C_0^2}{\omega_0^2} > 0. \tag{3.6}$$

The value

$$X = 0.5 \sum_{i=1}^{N} \sum_{j=1}^{N} (p_i q_j - p_j q_i)^2$$

does not change with time due to Eqs. (3.5); it is strictly related to the von Neumann entropy S_N of the oscillator ensemble [2] as

$$S_N = \ln\left(\sqrt{X - \frac{1}{4}}\right) + \sqrt{X}\,\mathrm{arg\,sinh}\left(\frac{\sqrt{X}}{X - \frac{1}{4}}\right).$$

The constancy of entropy suggests that the process of changing the system state due to a change in the oscillation frequency is adiabatic.

The constancy of X implies that for $t = \tau$

$$\overline{E}^2 = \left(X_0 \overline{\omega}^2 + \overline{L}^2 + \overline{C}^2\right).$$

There are no restrictions imposed on the control u, therefore in the problem of minimizing the final energy of the ensemble the value $\overline{\omega}$ can be taken to equal ω_1, and we can restate the problem as

$$\overline{E} = \sqrt{\omega_1^2 X_0 + \overline{L}^2 + \overline{C}^2} \longrightarrow \min \tag{3.7}$$

under the conditions (3.5). At the same time, $\overline{E}$ is obviously no less than $\omega_1\sqrt{X_0}$.

To solve the problem, we will change the variables in such a way that the right-hand parts of differential Eqs. (3.5) do not include unlimited control u. We denote new state variables by z_1, z_2, z_3:

$$z_1 = E + L, \quad z_2 = \frac{E - L}{\omega^2}, \quad z_3 = \frac{C^2}{\omega^2} \geqslant 0. \tag{3.8}$$

The original variables are related to the new ones as

$$E = 0.5(z_1 + \omega^2 z_2), \quad L = 0.5(z_1 - \omega^2 z_2), \quad C = \omega\sqrt{z_3}. \tag{3.9}$$

The invariant is equal to

$$X = X_0 = z_1 z_2 - z_3. \tag{3.10}$$

Thus, of the three new state variables only two are independent. The set of admissible states of the system on the plane z_1, z_2 definitely lies above the hyperbola $z_1 z_2 = X_0$.

The minimum possible value of the energy that can be achieved in arbitrarily long time with parametric control, and the corresponding values of the variables are equal to

$$\overline{E}^* = \omega_1\sqrt{X_0} = \overline{z_1}^*, \quad \overline{z_2}^* = \frac{\overline{z_1}^*}{\omega_1^2}. \tag{3.11}$$

We write Eqs. (3.5) for the new variables:

$$\frac{dz_1}{dt} = \frac{dE}{dt} + \frac{dL}{dt} = -2\omega C = -2\omega^2\sqrt{z_3}, \quad z_1(0) = z_{10}, \tag{3.12}$$

$$\frac{dz_2}{dt} = \frac{1}{\omega^2}\left(\frac{dE}{dt} - \frac{dL}{dt}\right) - \frac{2}{\omega}(E-L)u = 2\frac{C}{\omega} = 2\sqrt{z_3}, \quad z_2(0) = z_{20}. \tag{3.13}$$

In turn, $z_3 = z_1 z_2 - X_0$, so the system of equations for independent variables will take the form

$$\begin{cases} \dfrac{dz_1}{dt} = -2\omega^2\sqrt{z_1 z_2 - X_0}, & z_1(0) = z_{10} \\ \dfrac{dz_2}{dt} = 2\sqrt{z_1 z_2 - X_0}, & z_2(0) = z_{20}. \end{cases} \tag{3.14}$$

The initial state of this system is set, as well as the duration of the process. The problem is to find $\omega(t)$ so that $\overline{E}$ is minimal.

Since

$$\overline{L}^2 + \overline{C}^2 = \omega_1^2(\overline{z_1 z_2} - X_0) + 0.25(\overline{z_1} - \omega_1^2\overline{z_2})^2 = 0.25(\overline{z_1} + \omega_1^2\overline{z_2})^2 - \omega_1^2 X_0,$$

the optimality criterion can be written as

$$\overline{E} = 0.5(\overline{z_1} + \omega_1^2\overline{z_2}) \longrightarrow \min_{\omega_i \leqslant \omega \leqslant \omega_2}. \tag{3.15}$$

At the same time, for any constant control ω the phase trajectory on the plane z_1, z_2 is a straight line because

$$\frac{dz_1}{dz_2} = -\omega^2. \tag{3.16}$$

The right-hand side of Eq. (3.13) does not depend on ω and has the same sign as dz_2, so

$$dt = \frac{|dz_2|}{2\sqrt{z_1 z_2 - X_0}}. \tag{3.17}$$

Criterion (3.15) changes by virtue of Eqs. (3.14) to

$$\dot{E} = 0.5(\dot{z}_1 + \omega_1^2\dot{z}_2) = (\omega^2 - \omega_1^2)\sqrt{z_1 z_2 - X_0}. \tag{3.18}$$

The derivative is

$$\frac{dE}{dz_2} = -0.5(\omega^2 - \omega_1^2) \leqslant 0. \tag{3.19}$$

The problem of minimizing the final energy is transformed to the form

$$E_0 - \overline{E} = \int_{z_{20}}^{\overline{z_2}} (\omega^2 - \omega_1^2)dz_2 \longrightarrow \max_{\omega(z_2), \overline{z_2}} \tag{3.20}$$

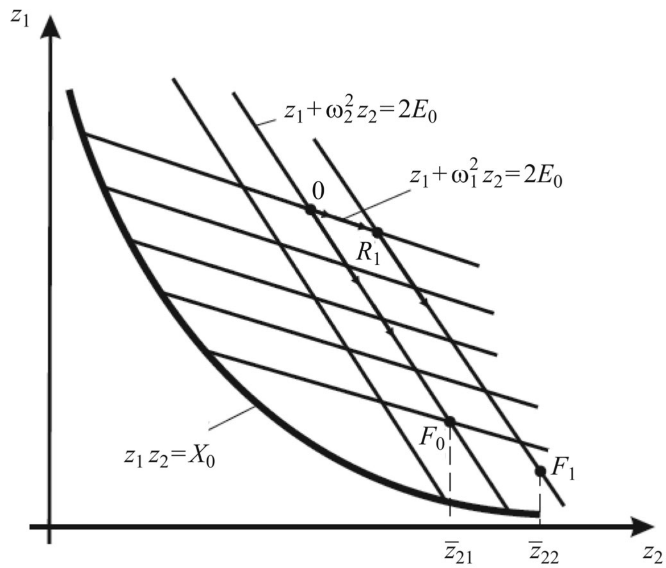

Fig. 2. Character of optimal processes on the plane z_1z_2.

under constraints (3.16) and

$$\int\limits_{z_{20}}^{\overline{z_2}} \frac{|dz_2|}{2\sqrt{(z_1(z_2)z_2 - X_0)}} = \tau. \tag{3.21}$$

Let υ denote the square of the oscillation frequency. The conditions of the maximum principle for problem (3.20), (3.21) will take the form ($\psi_0 = 1$)

$$H = \upsilon - \omega_1^2 + \frac{\lambda}{2\sqrt{z_1z_2 - X_0}} - \psi\upsilon, \tag{3.22}$$

$$\frac{d\psi}{dz_2} = -\frac{\partial H}{\partial z_1} = +\frac{\lambda z_2}{4(z_1z_2 - X_0)^{3/2}}, \quad \psi(\overline{z_2}) = 0. \tag{3.23}$$

Since with the growth of τ the value of $\Delta\overline{E}^* = E_0 - \overline{E}^*$ does not decrease (see [7]),

$$\lambda = -\frac{\partial\Delta\overline{E}^*}{\partial\tau} \leqslant 0. \tag{3.24}$$

Due to condition (3.23), $\frac{d\psi}{dz_2}$ does not change sign, which means that by virtue of condition (3.24) $\psi(z_2)$ decreases monotonically. Control $\upsilon^*(z_2) = \mathrm{sgn}(1 - \psi(z_2))$ delivers the maximum Hamiltonian function H. It is minimal (equal to ω_1^2) when $\psi(z_2) > 1$, and maximal (equal to ω_2^2) when $\psi(z_2) < 1$. Since $\psi(z_2)$ decreases monotonically, the control has at most one switching on the interval $z_{20} < z_2 < \overline{z}_2$.

Let the initial frequency be fixed and equal to ω_0 ($\omega_1 \leqslant \omega_0 \leqslant \omega_2$). The optimal frequency for $t = 0$ changes abruptly to ω_1 (if $\psi(z_{20}) > 1$) or to ω_2 (if $\psi(z_{20}) < 1$), then in any case at the end of the process it takes the value ω_2. At moment τ or, which is the same, for $z_2 = \overline{z_2}$, the control again abruptly decreases to ω_1^2.

When the system moves on the phase plane z_1, z_2 with any fixed frequency, its energy does not change, so it is easy to calculate the energy gain for any structure of the optimal process (Fig. 2).

Consider two possible structures of the optimal process.

1. *Process without switchings*. The frequency at the starting point 0 jumps from ω_0 to ω_2, the system moves with frequency ω_2 during the time τ to point F_0, where the frequency abruptly decreases to ω_1.

2. *Process with one switching*. The frequency at point 0 decreases from ω_0 to ω_1, the system moves with frequency ω_1 over time τ_{r1} to the switching point R_1, where it increases to ω_2. Then, with frequency ω_2 the system moves to the point F_1 over time $\tau - \tau_{r1}$, where the frequency abruptly drops to ω_1.

Let us compute the energy reduction for processes 1 and 2:

$$\Delta E_1 = E_0 - E_{F0} = (\omega_0^2 - \omega_2^2)z_{20} + (\omega_2^2 - \omega_1^2)\overline{z_{21}} = \omega_0^2 z_{20} + \omega_2^2(\overline{z_{21}} - z_{20}) - \omega_1^2\overline{z_{21}}, \tag{3.25}$$

$$\begin{aligned}\Delta E_2 = E_0 - E_{F1} &= (\omega_0^2 - \omega_1^2)z_{20} + (\omega_1^2 - \omega_2^2)z_{2R1} + (\omega_2^2 - \omega_1^2)\overline{z_{22}} \\ &= \omega_0^2 z_{20} + \omega_1^2(z_{2R1} - z_{20} - \overline{z_{22}}) + \omega_2^2(\overline{z_{22}} - z_{2R1}).\end{aligned} \tag{3.26}$$

Process 2 is preferable if $\delta = (\Delta E_2 - \Delta E_1) > 0$. The quantity δ is equal to

$$\begin{aligned}\delta &= \omega_1^2(z_{2R1} - z_{20} - \overline{z_{22}} + \overline{z_{21}}) + \omega_2^2(\overline{z_{22}} - z_{2R1} - \overline{z_{21}} + z_{20}) \\ &= (\omega_2^2 - \omega_1^2)(\overline{z_{22}} - z_{2R1} - \overline{z_{21}} + z_{20}).\end{aligned} \tag{3.27}$$

This value is positive if

$$\overline{z_{22}} - z_{2R1} > \overline{z_{21}} - z_{20}. \tag{3.28}$$

Thus, the process where the increment in the variable z_2 on the section where the frequency is equal to ω_2 is greater will be a better process. The switching process takes place in an area where the product $z_1 z_2$, and therefore also the rate of change of z_2 over time, is greater than for the process without switchings. On the other hand, the duration of this section is reduced due to the time needed to transit to the switching point. So the switching process is optimal, and the switching point must be selected according to the condition of maximizing ΔE.

The value $\overline{z_{21}}$ is determined by the condition (3.21):

$$\int\limits_{z_{20}}^{\overline{z_{21}}} \frac{dz_2}{2\sqrt{(2E_0 - \omega_2^2 z_2)z_2 - X_0}} = \tau, \tag{3.29}$$

which, after taking the integral, leads to an equation for $\overline{z_{21}}$

$$\arcsin\frac{E_0 - \omega_2^2\overline{z_{21}}}{\sqrt{E_0^2 - \omega_2^2 X_0}} = \arcsin\frac{E_0 - \omega_2^2 z_0}{\sqrt{E_0^2 - \omega_2^2 X_0}} - 2\omega_2\tau. \tag{3.30}$$

The variables in the left-hand side of inequality (3.28) are related to each other by the condition imposed on the duration of the process:

$$\int\limits_{z_{20}}^{z_{2R1}} \frac{dz_2}{2\sqrt{(2E_0 - \omega_1^2 z_2)z_2 - X_0}} + \int\limits_{z_{2R1}}^{\overline{z_{22}}} \frac{dz_2}{2\sqrt{(2E_0 - \omega_2^2 z_2)z_2 - X_0}} = \tau, \tag{3.31}$$

or, after taking integrals,

$$\begin{aligned}&\frac{1}{\omega_1}\left[\arcsin\frac{E_0 - \omega_1^2 z_{2R1}}{\sqrt{E_0^2 - \omega_1^2 X_0}} - \arcsin\frac{E_0 - \omega_1^2 z_{20}}{\sqrt{E_0^2 - \omega_1^2 X_0}}\right] \\ &+\frac{1}{\omega_2}\left[\arcsin\frac{E_0 - \omega_2^2\overline{z_{22}}}{\sqrt{E_0^2 - \omega_2^2 X_0}} - \arcsin\frac{E_0 - \omega_2^2 z_{2R1}}{\sqrt{E_0^2 - \omega_2^2 X_0}}\right] = 2\tau.\end{aligned} \tag{3.32}$$

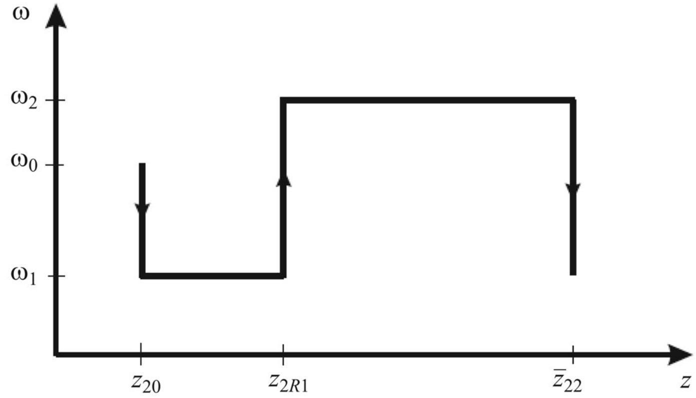

Fig. 3. Change of control parameter ω in the optimal process.

The choice of z_{2R1} reduces to finding the maximum difference $\overline{z_{22}} - z_{2R1}$ with condition (3.32). The optimality conditions for this problem, obtained using the Lagrange method, after eliminating an indefinite factor λ are of the form

$$F(\overline{z_{22}}, \omega_2) = F(\overline{z_{2R1}}, \omega_2) - F(\overline{z_{2R1}}, \omega_1), \tag{3.33}$$

where

$$F(\overline{z}, \omega) = \frac{1}{\sqrt{z(2E_0 - \omega^2) - X_0}}.$$

The system (3.32), (3.33) determines optimal values of $\overline{z_{2R1}}$ and $\overline{z_{22}}$.

The optimal process of changing the control parameter ω is shown in Fig. 3.

4. CONCLUSION

We have obtained the structures of optimal parametric control processes for a single oscillator and an ensemble of oscillators, allowing to minimize (increase) the oscillation energy over a given time τ; we have also obtained computational formulas for choosing the control switching moments. For a single oscillator, the oscillation energy can be made arbitrarily small over a sufficiently long time interval. For an ensemble of oscillators it can only be reduced to a certain limit. The reason for this difference is that an oscillator ensemble is a macrosystem where the control changes the oscillation frequency common to the ensemble and does not affect their individual phases. In this regard, the capabilities of "adiabatic" control are limited.

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