# Heat exchange systems with minimal irreversibility

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ABSTRACT

The conditions for minimum dissipation and the dependence of the minimum possible entropy production on the heat load are obtained for heat exchange systems with a given heat load and total heat transfer coefficient. The case of variable heat capacity of streams is considered. Distributions of heat exchange coefficients and heat capacity rates of the streams for which this minimum of irreversibility can be achieved are found. The problem is solved for a wide class of heat transfer kinetic laws in which the heat flow is proportional to the contact surface. The possibility of evaporation and condensation of streams is investigated. The concept of thermodynamic equivalence of dual-stream and multi-stream systems is introduced and the conditions that lead to such equivalence are obtained. Using this equivalence for Newtonian kinetics, optimal synthesis conditions are obtained for dual-stream and multi-stream systems with a given total heat transfer coefficient and heat load. It is shown how the solution can be implemented in a system of dual-stream heat exchangers.

## 1. Introduction

The thermodynamic approach to the analysis of technological systems (see e.g. Shnip, 2018; Kakac, 2018) allows one to identify the boundary of the set of realizable systems and to find their maximum capabilities (heat and mass transfer systems, heat and refrigeration machines, separation systems, chemical reactors, etc.). Most of these bounds are based on the relations of thermodynamics of reversible processes (Carnot efficiency, reversible work of Gibbs separation) and they are very important as ultimate limits. However, they are quite optimistic in comparison with the capabilities of real systems. The reason for this is that these reversible models do not take into account the intensity of flows, the size of the contact surfaces, and other factors related to the finite dimensions of the devices and required speeds of operation.

Attempts to take these factors into account and obtain the characteristics of heat and refrigeration machines that are optimal in the class of irreversible processes have led to the intensive development of optimization thermodynamics or finite-time thermodynamics (Andresen, 1983; Berry et al., 1999). Thus, along with the ideal reversible Carnot heat engine, the maximum power (Curzon-Ahlborn) heat engine (Novikov, 1958; Curzon and Ahlborn, 1975), as well as a fixed power heat engine with maximum efficiency were considered (Rozonoer and Tsirlin, 1983a,b,c). In all these cases, we solved the prob-

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**Nomenclature**

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| $C$ | specific heat capacity, J/kg K |
| $F$ | contact surface area from system inlet, m$^2$ |
| $k$ | local heat transfer coefficient, its dimension depends on the dimension of $z$ |
| $K$ | total coefficient of heat exchange, W/K |
| $g$ | mass flow evaporated/condensed, kg/s |
| $r$ | specific heat of vaporization/condensation, J/kg |
| $\sigma$ | entropy production rate, W/K |
| $T_+, T_-$ | temperature of hot and cold streams at a particular point, K |
| $q$ | local heat flow intensity, W/m$^2$ |
| $Q$ | heat load, W |
| $W$ | heat capacity rate of stream, W/K |
| $z(T_+, T_-)$ | temperature multiplier ("heat transfer law") |
| $b$ | subscript indicating 'boiling' |
| $+, -, *, i, j$ | indices |
| $I(Q), J(Q)$ | shorthand for intermediate quantities |

lem of an ideal heat engine with natural restrictions on the heat exchange coefficients of the working medium with its heat sources and on the power of the machine. We emphasize that other restrictions can be taken into account as well such as the amount of friction between the piston of the heat engine and the cylinder surface, heat loss to the environment, etc. (see e.g. Andresen and Huleihil, 2006).

For some processes, reversible bounds only make sense as unrealistic idealizations. One of the most important examples of such systems is heat exchange systems. The assessment of thermodynamic perfection (how close they are to reversible performance) of these is impossible without taking into account the limited contact surface (the total coefficient of heat exchange) and the required heat load (the amount of heat transferred per unit of time from hot to cold streams). To assess the perfection of such systems, an exergetic approach is used (see e.g. Brodjanskiy et al., 1988; Bosnjakovic, 1965), comparing the systems by the loss of exergy in each of them.

Exergy losses are proportional to the entropy production in the system and the ambient temperature $T_0$. In the exergetic analysis of a heat exchange system, the entropy production in the system is calculated using the known heat capacities of the streams and the temperatures of the streams at the inputs and outputs of the system without going into detail about how to construct a mechanism to do this, i.e. without solving the problem of system synthesis. The synthesis problem is defined as follows: *Which conditions must be met by the heat exchange system so that the production of entropy in it (and hence the loss of exergy) under the given restrictions is minimal.* This paper is devoted to solving this problem.

A significant number of studies have been devoted to the conditions for minimum dissipation in thermodynamic processes. Thus, in their celebrated work (Tondeur and Kvaalen, 1987) Tondeur and Kvaalen propose constancy of the rate of entropy production ("equipartition of entropy production") as a condition for achieving minimal irreversibility for a certain class of systems, primarily with linear rate dependence. While this is a wonderfully simple conclusion which holds in many practical situations and is a good approximation in many others, it is frequently used indiscriminately also outside its region of applicability. In Tsirlin et al. (1998) derive conditions for minimum dissipation for arbitrary kinetics. And finally in Tsirlin et al. (2003) it is proved that minimum dissipation corresponds to constancy of the rate of entropy production only under certain conditions, and a class of kinetic regularities is distinguished for which this condition is valid. To use the results of these studies in our problem it was necessary to take into account the restrictions on the temperatures of all parts of the streams, changes in their heat capacities from temperatures, etc.

Another celebrated approach to minimizing dissipation in multi-stream heat exchange systems is pinch analysis (Linnhoff and Hindmarsh, 1983) (and possibly textbooks like Smith, 2005; Kemp, 2006). The method is based on aggregated heat loads of the entire system at individual temperatures arranged in $(T, Q)$ plots analogous to e.g. our Fig. 3. As a result of this construction, the point where the temperature difference between heat supply and heat demand is minimal (the pinch point) is determined, and the desired structure of the system may de be concluded. This is typically obtained with given heat loads and given constant heat conductances for each element of the system. By contrast, our derivation below has at its disposal a total amount of heat conductance which may be distributed among the heat exchange units for total optimality. Further, we consider a wide range of heat conductance dependencies as well as include the possibility of phase change (boiling and/or condensation). It follows from relations derived below that the vicinity of the pinch point makes the maximum contribution both to the production of entropy and to the value of the required heat transfer coefficient. Thus the relations derived here can be viewed as the thermodynamic justification for pinch analysis.

This paper consists of two sections. The first is devoted to obtaining conditions for minimum dissipation in dual-stream systems for various forms of heat transfer dependencies and for fixed streams with variable heat capacity and with a given temperature change profile. These results are used in the second section where multi-stream systems are considered and conditions for thermodynamic equivalence of dual-stream and multi-stream systems are obtained. This equivalence makes it possible to implement multi-stream systems with minimal irreversibility using the structure of dual-stream heat exchangers.

## 2. System synthesis

Minimizing entropy production leads to systems that, under given constraints, can use hot streams with a minimum temperature, or heat cold ones to the maximum possible temperature. Indeed, the entropy production, as is known (Bosnjakovic, 1965), is equal to the product of the heat flow and the driving force, so that for a given flow, the minimum of irreversibility corresponds to the minimum of the driving force.

For heat transfer kinetics of the form $q = k(T_+ - T_-)$ (Newtonian), the *optimal heat transfer system with minimal entropy production rate $\sigma^*$ for a given total heat conductance $\overline{K}$ between hot and cold streams and total heat load (heat transfer rate between the two streams) $\overline{Q}$* was derived in Tsirlin (2017). Along with the total heat transfer coefficient and total heat flux, were assumed known characteristics of the part streams: Their heat capacity rates (equal to the product of flow rate times specific heat capacity $W_i = g_i C_i$, typically measured in W/K) and temperatures at the inlet to and outlet from the system $T_i^0$.

In this paper the lower bound of irreversibility is found for a wide class of heat transfer functions $q$ with the ability to determine not only the heat capacity rates and input temperatures, but also the profiles of temperature changes of some streams and their variable heat capacities. Under these conditions, formulas are obtained for calculating the temperatures of the remaining streams and the distribution of heat transfer coefficients corresponding to the minimum dissipation along the contact course.

The conditions of minimum dissipation which are valid for variable heat capacities of streams made it possible to consider an important class of heat exchange systems in which part of the streams condense or evaporate. In turn, these conditions allowed converting a multi-stream heat exchange system into a thermodynamically equivalent system with two streams. This equivalence is used for the synthesis of multi-stream systems below.

This analysis of the minimum entropy production rate

- shows how one or another of the specified factors affect the system's capabilities. Those factors could be temperatures and heat capacity rates of fixed flows, heat load, heat transfer coefficient, etc.;
- allows us to estimate the thermodynamic efficiency of a real heat exchange system by comparing the actual entropy production with the minimum possible;
- allows us to find the boundary of the system's realizability. Indeed, if in the design process the requirements for a proposed heat exchange system are such that the entropy production in it is less than in the optimal system, then the system is not physically achievable, just as a theoretical heat engine whose efficiency exceeds the Carnot efficiency is impossible.

We will call the streams for which the temperatures and heat capacities are set *fixed*, and the streams whose parameters need to be found under the minimum dissipation conditions *controlled*. In the problem with certain conditions imposed on the fixed streams one needs to select the parameters of controlled streams so that entropy production is minimal.

## 3. Minimum dissipation conditions for dual-stream heat exchange systems

Let us limit ourselves to the dependence of heat flow on the temperatures of heat carriers in the form

$$q(T_+, T_-) = kz(T_+, T_-), \tag{1}$$

where $k$ is the local heat conductance, and $z(T_+, T_-)$ is the temperature multiplier, i.e. the dependence of the heat transfer rate on the temperatures of the two sides of the heat exchanger at that point, $T_+$ on the hot side, $T_-$ on the cold side, independent of the size of the equipment. This form includes many well-known laws of heat transfer, such as Newton, Fourier, and radiant heat transfer:

$$q(T_+, T_-) = k(T_+ - T_-) \tag{2}$$

$$q(T_+, T_-) = k(1/T_- - 1/T_+) \tag{3}$$

$$q(T_+, T_-) = k(T_+^4 - T_-^4). \tag{4}$$

The heat flow $q(T_+, T_-)$ refers to the unit of contact surface $F$ and has a dimension of W/m$^2$. However, this does not include kinetics of the additive form $q(T_+, T_-) = k_1 z_1(T_+, T_-) + k_2 z_2(T_+, T_-)$.

We assume that the temperature multiplier $z(T_+, T_-)$ satisfies the following natural conditions:

$$T_+ > T_-, \quad q > 0, \quad \frac{\partial z}{\partial T_+} > 0, \quad \frac{\partial z}{\partial T_-} < 0.$$

Temperatures of the contacting streams vary monotonically with progressing area of contact $F$ (Fig. 1). $F$ takes values from 0 to a total area $\overline{F}$ of the heat exchanger. With increasing $F$ also the heat flow $Q$ (the heat load) transferred from hot to cold stream increases.

This value changes from zero at $F = 0$ to $Q(\overline{F}) = \overline{Q}$, the total heat load. The heat transfer coefficient $K$ which varies from 0 to $\overline{K} = K(\overline{F})$ also depends monotonically on $F$. Any quantity that depends monotonically on $F$ can be taken as the independent variable in the description of the heat transfer. The analysis of the system is greatly simplified if we take the heat load $Q$ as that variable. Moreover, the local heat flow

$$q = kz(T_+, T_-) = \frac{dQ}{dF}. \tag{5}$$

The coefficient $k$ in the formula (5) is equal to $k = \frac{dK}{dF}$. Its dimension depends on the dimension of the temperature factor $z(T_+, T_-)$.

Let us focus on stationary systems and later show what changes in the obtained relations for periodic heat exchange processes.

To calculate the heat flow along the contact surface, one can use other variables than the heat exchange area $F$ as an argument as long as they increase monotonically with $F$. Such a variable may e.g. be the temperature of the cold heat carrier $T_-(F)$. In this case $T_-(0) = T_-^{in}$ and $T_-(\overline{F}) = T_-^{out}$.

Another such variable is the intermediate heat load,

$$Q(F) = \int_0^F q(T_+, T_-) dF. \tag{6}$$

Since the *heat transfer function* $q > 0$ everywhere, $Q(F)$ increases monotonically. This variable will be called *heat load* for short. It is the heat flow transmitted to a cold heat carrier when its temperature rises from the initial $T_-^{in}$ to some intermediate $T_-(F)$.

The dimension of the heat transfer function depends on what variable is selected as the argument. If it is the contact surface $F$, then the dimension of the heat transfer function is W/m$^2$. If the argument is temperature, then the dimension $q$ is W/K.

Another variable that monotonically increases with the growth of the contact surface is the heat transfer coefficient

$$K(F) = \int_0^F k(F) dF = \int_0^F \frac{q(T_+, T_-)}{z(T_+, T_-)} dF \quad \text{or} \quad K(Q)$$

$$= \int_0^Q \frac{dQ}{z(T_+, T_-)}, \tag{7}$$

where $k(F)$ is the coefficient in the heat transfer function $q(T_+, T_-)$. The dimension of $k$ depends on the dimensions of the heat transfer function and the temperature multiplier $z(T_+, T_-)$.
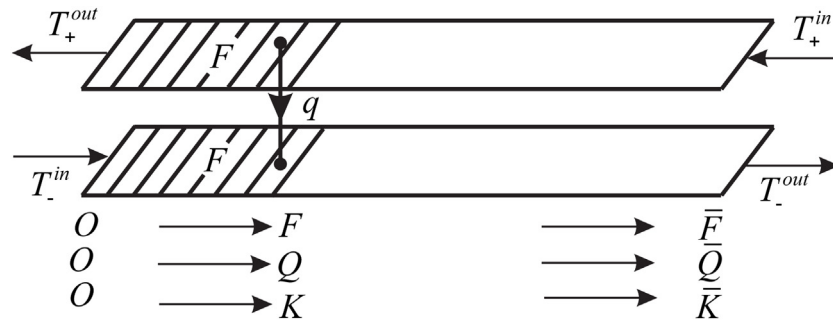
**Fig. 1 – The illustration of the transition from the area of contact $F$ to the heat flow $Q$ (the heat load) and the heat transfer coefficient $K$, that monotonically depend on $F$.**

The solution of the problems discussed below is significantly simplified if the heat load $Q$ is selected as the independent variable. In this case

$$\frac{dQ}{dT_-} = \frac{1}{W_-}, \quad \frac{dQ}{dT_+} = \frac{1}{W_+}. \tag{8}$$

If a counterstream is implemented in the system, then in the last equality you need to change the minus sign to a plus, while the temperature of the hot stream increases with the growth of $Q$.

When using this notation, the problem statement about the minimum dissipation takes the form:

$$\sigma = \int_0^{\overline{F}} kz(T_+, T_-) \left( \frac{1}{T_-} - \frac{1}{T_+} \right) dF \to \min / \int_0^{\overline{F}} k(F) dF = \overline{K}, \tag{9}$$

$$\int_0^{\overline{F}} kz(T_+, T_-) dF = \overline{Q}.$$

At the same time

$$\frac{dT_+}{dF} = -\frac{kz(T_+, T_-)}{W_+}, \quad \frac{dT_-}{dF} = \frac{kz(T_+, T_-)}{W_-}. \tag{10}$$

Here $W_+$, $W_-$ are the heat capacity rates, equal to the product of their stream flow rates and specific heat capacities. If the contact mode is counter-current, then the minus sign in the first equality should be replaced with a plus and the slope of both curves will be positive.

We transform the problem by changing the argument from the contact surface $F$ to the heat load $Q$, which will significantly simplify its solution. Such a transition is possible because the positive heat exchange flow $q$ makes $F$ and $Q$ depend monotonically on each other.

Let us use the equality (7), expressing $dF$ in terms of $dQ$ and substitute it into the problem conditions (9). Rewrite the converted task for a counter-current flow:

$$\sigma = \int_0^{\overline{Q}} \left( \frac{1}{T_-} - \frac{1}{T_+} \right) dQ \to \min / \int_0^{\overline{Q}} \frac{dQ}{z(T_+, T_-)} = \overline{K}, \tag{11}$$

$$\frac{dT_+}{dQ} = \frac{1}{W_+(Q)}, \quad \frac{dT_-}{dQ} = \frac{1}{W_-(Q)}. \tag{12}$$

The last condition in (9) for this transition is taken into account by selecting the upper limit of integration in the converted problem, and the conditions (12) can be discarded by taking the temperature of the controlled stream as a control. Temperature profiles of contacting streams through condi-

tions (12) are associated with corresponding changes in heat capacity rates. To be definite, we will initially consider the cold stream fixed and select $T_+(Q)$ in the problem (11). This stream can have a variable heat capacity rate $W_-(Q)$ which determines the dependence $T_-(Q)$ using Eq. (12).

We use the Lagrange method to solve the problem (11). Assuming a non-degenerate solution, the Lagrange function of the problem (11) has the form:

$$L = \left( \frac{1}{T_-} - \frac{1}{T_+} \right) + \frac{\lambda}{z(T_+, T_-)}. \tag{13}$$

The conditions of its stationarity in $T_+$ lead to the equality

$$\frac{z^2(T_+, T_-)}{T_+^2 \partial z/\partial T_+} = \lambda. \tag{14}$$

This condition determines the temperature of the controlled stream through the temperature of the fixed stream to within the constant $\lambda$ which is found by substituting the solution into the integral constraint of the problem (11).

In the case when the fixed stream is hot, the condition (14) will take the form:

$$\frac{z^2(T_+, T_-)}{T_-^2 \partial z/\partial T_-} = -\lambda. \tag{15}$$

As an example, we will find a solution to the problem and the corresponding minimal irreversibility for the heat transfer laws mentioned above (2).

### 3.1. Newtonian kinetics

The condition (14) leads to a known result (see Tsirlin, 2017, 1991; Andresen and Gordon, 1992):

$$\frac{T_+ - T_-}{T_+} = \sqrt{\lambda} \quad \Rightarrow \quad \frac{T_-}{T_+} = const = m < 1. \tag{16}$$

This condition can be fulfilled if one organizes the heat exchange process in counter-current mode (in this case, the temperatures of both streams increase with the increase of $Q$), choosing the initial temperature of the hot stream equal to $T_+^0 = \frac{T_-^0}{m}$, and the heat capacity rate $W_+(Q)$ equal to $W_-(Q)m$.

To find $m$ we substitute the equality (16) into the integral constraint for the heat transfer coefficient. We obtain

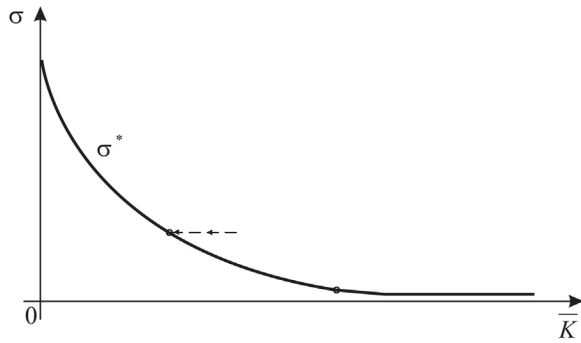$$\int_0^{\overline{Q}} \frac{1}{T_-(Q)(1/m - 1)} dQ = \overline{K}. \tag{17}$$

**Fig. 2 – The dependence of the minimum entropy production $\sigma$ on the heat transfer coefficient $\overline{K}$.**



**Fig. 3 – The dependencies of the flow temperatures on the thermal load during the evaporation of the cold coolant.**

The solution of this equation with respect to the constant $m$ will take the form:

$$m = \frac{\overline{K}}{I(\overline{Q}) + \overline{K}}, \tag{18}$$

where for short we define $I(\overline{Q})$ as

$$I(\overline{Q}) = \int_0^{\overline{Q}} \frac{dQ}{T_-(Q)}. \tag{19}$$

The thermal conductance and the minimum entropy production for optimal Newtonian heat exchange depend on the temperature of the fixed stream as:

$$k^*(Q) = \frac{m}{T_-(Q)(1-m)} = \frac{\overline{K}}{I(\overline{Q})T_-(Q)}, \quad \sigma^*$$

$$= I(\overline{Q})(1-m) = \frac{I^2(\overline{Q})}{I(\overline{Q}) + \overline{K}}. \tag{20}$$

The star indicates optimality. Note that the authors of Tsirlin (2017, 1991) consider only streams with constant heat capacity rates, not streams with different heat capacity rates at different temperatures, arising e.g. from different heating demands at these temperatures (different slopes in Fig. 3) or from condensation/evaporation (horizontal segments in Fig. 3).

It follows from (20) that the dependence of the minimum entropy production on the heat transfer coefficient is a monotonic and downward convex function (Fig. 2).

The solid line is the realizability boundary for all heat transfer systems having a given heat load $\overline{Q}$. Each heat transfer system corresponds to a display point lying above and to the right of this border. An improvement to this system, in which the dissipation in it with the same $\overline{Q}$ value becomes minimal, corresponds to the movement of the mapping point to the realizability boundary (see the dashed arrow). In this case, the heat transfer coefficient $\overline{K}$ decreases. So for a given heat load, heat transfer with minimal dissipation also corresponds to a minimum heat transfer coefficient.

It is the boundary of physically implemented dual-stream heat exchange systems. The proximity of the characteristics of the heat exchange system to this boundary characterizes its thermodynamic perfection. In this case, for a given total heat load $\overline{Q}$, the minimum heat transfer coefficient corresponds to a point lying on the border of realizability, i.e. a system operated at minimum dissipation.
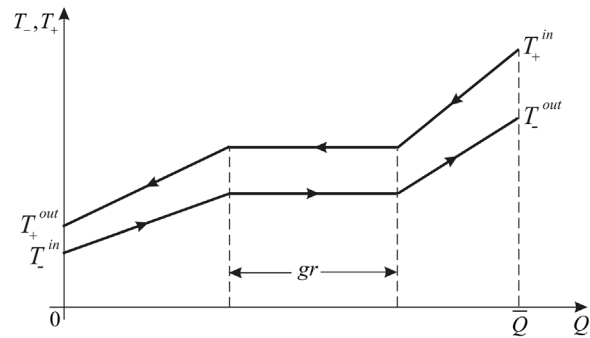
In the case when a fixed stream is hot, the relations (18), (19) will take the form:

$$m = 1 - \frac{J(\overline{Q})}{\overline{K}}, \quad J(\overline{Q}) = \int_0^{\overline{Q}} \frac{dQ}{T_+(Q)}, \tag{21}$$

where again we define an intermediate quantity $J(\overline{Q})$. Since $0 < m < 1$ and $\overline{K} > J(\overline{Q})$, the optimal conductance and minimum entropy production are

$$k^*(Q) = \frac{\overline{K}}{J(\overline{Q})T_+(Q)}, \quad \sigma^* = J(\overline{Q})\frac{1-m}{m} = \frac{J^2(\overline{Q})}{\overline{K} - J(\overline{Q})}. \tag{22}$$

### 3.1.1. Fixed stream with phase state change

An important special case of using the obtained relations is the calculation of the parameters of a dual-stream system in which a fixed (for concreteness: cold) stream enters as a liquid and leaves as superheated vapor.

For the cold stream, the flow rate $g$, the specific heat capacities $C_1$ and $C_2$ of liquid and vapor, the specific heat of vaporization $r$, the boiling point $T_b$, the initial and final temperatures $T_-^{in}$, $T_-^{out}$, the value of the integral heat exchange coefficient $\overline{K}$, and the specific heat exchange coefficients at the stages of liquid heating, boiling, and superheating, $k_1^0, k_2^0, k_3^0$ are specified. For most liquids the inequalities $k_2^0 > k_1^0 > k_3^0$ are valid (i.e. the heat transfer coefficient for a saturated vapor is bigger than for its liquid and still bigger for superheated vapor). These values completely determine the dependence $T_-(Q)$ and the total heat load $\overline{Q}$.

The sequence of calculations is as follows:

1. The heat capacity rates of the cold stream at the stages of liquid heating and vapor overheating, $W_{-1} = gC_1$, $W_{-2} = gC_2$. At the evaporation stage the heat capacity is infinite and the required heat flow is equal to $gr$.
2. Heat load. At the liquid heating stage $Q_1 = W_{-1}(T_b - T_-^{in})$. Total heat load $\overline{Q} = Q_1 + gr + W_{-2}(T_-^{out} - T_b)$. The found dependence $T_-(Q)$ is shown in Fig. 3(lower curve).
3. To calculate the value of $m$ and the value of the change in the heat transfer coefficient, we find the minimum entropy production using the formulas (18) and (20) in which one needs to use the specified variables to express $I(\overline{Q})$. This integral is not difficult to calculate as the sum of integrals in each of the three stages. The result is

$$I(\overline{Q}) = I_1 + I_2 + I_3 = W_{-1} \ln \frac{T_b}{T_-^{in}} + \frac{gr}{T_b} + W_{-2} \ln \frac{T_-^{out}}{T_b}. \tag{23}$$

The minimum entropy production $\sigma^*$ is determined by the expression (20) after substituting $I(\overline{Q})$ into it.

4. Implementation and calculation of contact surfaces. The heat exchange system consists of three stages in each of which the heat capacity rate of the cold stream is constant and can be implemented in the form of three dual-stream heat exchangers. In the first of these the liquid is heated, in the second it is vaporized, and in the third it is superheated as vapor. The contact surfaces for each of these stages are determined by the relations:

$$F_1 = \frac{\overline{K}I_1}{I(\overline{Q})k_1^0}, \quad F_2 = \frac{\overline{K}I_2}{I(\overline{Q})k_2^0}, \quad F_3 = \frac{\overline{K}I_3}{I(\overline{Q})k_3^0}. \tag{24}$$

The first and third stages are counter-current and the heat capacity rate of the hot stream should be less than the heat capacity rate of the cold stream by the factor $m$. In the evaporation stage of a cold stream ideally the hot stream should condense at a temperature of $\frac{T_b}{m}$. In case of an approximate implementation of a heat exchange system, it is necessary to maximize the heat capacity rate of the hot stream at this stage, taking into account the existing restrictions in the system.

### 3.2. Other forms of heat transfer equations

Let us briefly discuss the calculated relations for other forms of heat flow dependence on contact temperatures.

*Fourier kinetics.* Conditions (14) for a fixed cold stream lead to the relations:

$$\frac{1}{T_-} - \frac{1}{T_+} = const = M, \quad M = \frac{\overline{Q}}{\overline{K}}, \quad k^*(Q) = \frac{\overline{K}}{\overline{Q}}, \quad \sigma^* = \frac{\overline{Q}^2}{\overline{K}}, \tag{25}$$

$$T_+^*(Q) = \frac{\overline{K}T_-(Q)}{\overline{K} - \overline{Q}T_-(Q)}.$$

After differentiating the last equality we get the relation of the heat capacity rate of the hot stream with the specified parameters of the system:

$$W_+^*(Q) = \frac{W_-(Q)}{\overline{K}^2}(\overline{K} - \overline{Q}T_-(Q))^2. \tag{26}$$

For Fourier kinetics, the local thermal conductivity coefficient $k$ is constant, and the dependence of the minimum achievable entropy production on the thermal conductivity coefficient has the form of a hyperbola.

*Radiant heat exchange.* Conditions (14) for a fixed cold stream lead to a relation between the temperatures of the controlled and fixed streams of the form

$$T_+^3 + \frac{T_-^8}{T_+^5} - 2\frac{T_-^4}{T_+} = const. \tag{27}$$

The value of this constant, defined by the values $\overline{Q}$ and $\overline{K}$, can only be found numerically.

For all types of heat transfer kinetics on the optimal solution, the heat transfer coefficient $k$ and the temperature multiplier $z$ depend on the current heat load $Q$ so that their product $q$ is constant.

For a periodic process, all the results obtained are valid with the contact surface $F$ replaced by the process duration $\tau$ varying from zero to the specified duration $\overline{\tau}$.

## 4. Multi-stream heat exchange systems

### 4.1. Equivalent streams and minimal dissipation conditions

Consider a multi-stream system consisting of $n$ cold and $p$ hot streams with temperatures $T_{j-}$ and $T_{i+}$ and obtain conditions for minimal heat exchange dissipation for it. In systems integrated with technological processes as a rule most restrictions are imposed on the cold streams, so to be specific let us consider all cold streams fixed while the parameters of the hot streams will be selected according to the condition of minimum exergy dissipation (or equivalently, minimum entropy productions).

Now we introduce the concept of *equivalent stream*: *An equivalent cold (hot) stream is a stream whose heat capacity rate at each temperature is equal to the sum of the heat capacity rates of all cold (hot) streams having the same temperature.*

Thus, the heat capacity rate of an equivalent stream can change with temperature due to changes in the number of component streams whose temperatures at the input and output of the system are different. If at least one component stream changes its phase, then the heat capacity rate of the equivalent stream is infinitely large in this section which means that the dependence of its temperature on the current heat load has a horizontal section.

Definition: *Two heat exchange systems are thermodynamically equivalent if they have the same inlet and outlet stream temperatures, total heat loads $\overline{Q}$, total thermal conductivity coefficients $\overline{K}$, and entropy production $\sigma$.*

Using this definition we

- consider for calculational purposes a dual-stream heat transfer system in which the cold stream has a variable heat capacity rate and is equivalent to the cold streams of the multi-stream system being synthesized;
- write down the conditions under which the designed system is thermodynamically equivalent to a system consisting of dual-stream heat exchangers, each of which receives one cold stream;
- solve the problem of minimal dissipation for the calculational dual-stream system;
- select the parameters of a multi-stream system (heat loads, heat transfer coefficients, heat capacity rates of controlled streams) so that it is thermodynamically equivalent to the calculated dual-stream system with minimal dissipation.

*Temperature change of the equivalent calculational cold stream.* According to the definition above, the temperature of the equivalent calculational cold stream $T_-(Q)$ satisfies the conditions:

$$T_-^{in} = \min_i T_{i-}^{in}, \quad T_-^{out} = \max_i T_{i-}^{out}, \quad \frac{dT_-}{dQ} = \frac{1}{\sum_i W_{i-}(T_{i-})}. \tag{28}$$

In this expression the heat capacity rates of the synthesized cold streams $W_{i-}(T_{i-}) = \frac{1}{dT_{i-}/dQ}$ are zero for $T_{i-} < T_{i-}^{in}$ and for $T_{i-} > T_{i-}^{out}$. This corresponds to the fact that outside the interval $[T_{i-}^{in}; T_{i-}^{out}]$ there is no heat transfer, since there is no stream.

The dependence $T_-(Q)$ is a continuous piecewise linear function on the areas of constant heat capacity $Q$. At each of the break points the temperature $T_-(Q)$ is either equal to the boiling point of one of the streams or the temperature at the

inlet or outlet of one of the cold streams. In the latter case the abscissae of the break points are denoted $Q_i^{in}$ and $Q_i^{out}$.

**Theorem 1.** *For any law of change in the temperature of the hot stream $T_+(Q) > T_-(Q)$, the calculated two-stream system is thermodynamically equivalent to a multi-stream system consisting of n heat exchangers, in each of which the ith cold stream contacts the ith hot stream under the following conditions:*

(29) *The temperature of the hot stream $T_{i+}(Q)$ in each i heat exchanger coincides with the temperature $T_+(Q)$ on the contact interval $Q_i^{in} \leq Q \leq Q_i^{out}$. This means that the temperature of the hot stream at the inlet to the i heat exchanger should be equal to the temperature $T_+(Q_i^{out})$, and the heat capacity rates $W_i$ for the period of contact should relate to each other as well as to the equivalent heat flows according to*

$$\frac{W_{i+}}{W_{i-}} = \frac{dT_{i-}}{dT_{i+}} = \frac{dT_-}{dT_+} = \frac{W_+}{W_-}. \tag{29}$$

(30) *Each heat exchanger is countercurrent with a full heat load as specified for the problem conditions:*

$$\bar{Q}_i = \int_{T_{i-}^{in}}^{T_{i-}^{out}} W_{i-}(T_{i-})dT_{i-} + g_{i-}r_{i-}. \tag{30}$$

**Proof.** The total heat load under the assumption that the pressure in the system changes little is equal to:

$$\overline{Q} = \int_{T_-^{in}}^{T_-^{out}} \sum_i W_{i-}(T_{i-})dT_{i-} + \sum_i g_{i-}r_{i-} = \sum_i \overline{Q}_i, \tag{31}$$

since $dT_- = dT_{i-}$. If the ith stream does not evaporate the summand $g_{i-}r_{i-}$ is zero.

Temperature profiles in the heat exchangers and in the calculation system are identical, and therefore the temperature multipliers are the same ($z_i(T_{i+}, T_{i-}) = z(T_+, T_-)$), so that the total heat transfer coefficient according to (11) is equal to

$$\overline{K} = \int_0^{\overline{Q}} \frac{dQ}{z(T_+, T_-)} = \int_0^{\overline{Q}} \frac{\sum_i dQ_i}{z(T_+, T_-)} = \sum_i \int_{Q_i^{in}}^{Q_i^{out}} \frac{dQ_i}{z_i(T_{i+}, T_{i-})}$$

$$= \sum_i \overline{K}_i. \tag{32}$$

Here $dQ_i = dQ \frac{W_{i-}}{W_-}$.

Finally, the entropy production due to the coincidence of temperature profiles in the heat exchangers and in the calculation system on the contact intervals of the streams can be expressed as

$$\sigma = \int_0^{\overline{Q}} \frac{dQ}{1/T_- - 1/T_+} = \int_0^{\overline{Q}} \frac{\sum_i dQ_i}{1/T_- - 1/T_+}$$

$$= \sum_i \int_0^{\overline{Q}_i} \frac{dQ_i}{1/T_- - 1/T_+} = \sum_i \sigma_i. \tag{33}$$

Thus, the statement of thermodynamic equivalence is proved.

A consequence of this theorem is that *if a hot stream temperature profile is found for which the dissipation in the calculation system is minimal and the conditions of thermodynamic equivalence*
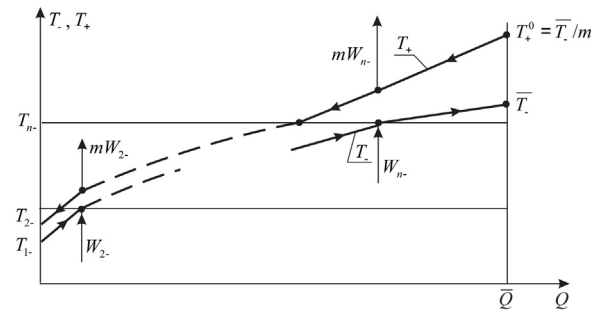


**Fig. 4 – Dependence of the equivalent stream temperatures on the current heat load.**

(29) and (30) *are met in the multi-stream system, then the entropy production in this latter system is also minimal.*

Fig. 4 shows the nature of the dependence of the temperatures of equivalent streams on the heat load. The vertical arrows indicate the points of changing the composition of equivalent streams, and next to the arrows are indicated the entering and leaving heat capacity rates of the system streams. For any value of the heat load $Q$, the ratio of temperatures of the equivalent streams $\frac{T_-}{T_+}$ is constant and equal to $m < 1$ and the ratio of their heat capacity rates $\frac{W_+}{W_-}$ are equal to $\frac{1}{m}$.

The problem of selecting a hot (controlled) stream profile for any fixed dependence of the cold stream temperature on the heat load is solved in the first part of the work. Its solution determines the optimal temperature $T_+(Q)$ and hence the optimal heat capacity rate $W_+(Q) = dT_+/dQ$ of the equivalent hot stream. The solution type is shown in the Fig. 4.

It remains to figure out how to choose the heat capacity rates of hot streams and heat transfer coefficients for each of the $n$ heat exchangers in a multi-stream system.

For each ith heat exchanger, the cold stream parameters are known and its heat load $\overline{Q}_i$ is defined by the expression (30).

By the equivalence condition (29) the heat capacity rate of the hot stream for each value $Q$ is equal to

$$W_{i+} = W_+ \frac{W_{i-}}{W_-}. \tag{34}$$

If the cold stream evaporates at $T_{ib-} = T_-$, the hot stream must condense at $T_{ib+} = T_+$. In this case the corresponding amount is determined by the energy balance $g_{i+}r_{i+} = g_{i-}r_{i-}$.

The heat conductivity coefficient of the ith heat exchanger $\overline{K}_i$ is calculated by (32) as

$$\overline{K}_i = \int_{Q_i^{in}}^{Q_i^{out}} \frac{W_{i-}dQ}{W_- z(T_+, T_-)}. \tag{35}$$

Let us illustrate the possibilities of using the obtained relations on the example of the synthesis of a system with three fixed streams and Newtonian heat transfer kinetics.

### 4.2. Example

Consider a system with three cold streams with the following characteristics (Fig. 5):

The first stream enters in the form of a liquid with an initial temperature of $T_{1-}^{in} = 300\,K$ and a heat capacity rate of $W_{1l-} = 4\,kW/K$. It boils at a temperature of $T_{b-} = 370\,K$ consuming heat at the rate $g_{1-}r_{1-} = 1000\,kW$. The vapor is then
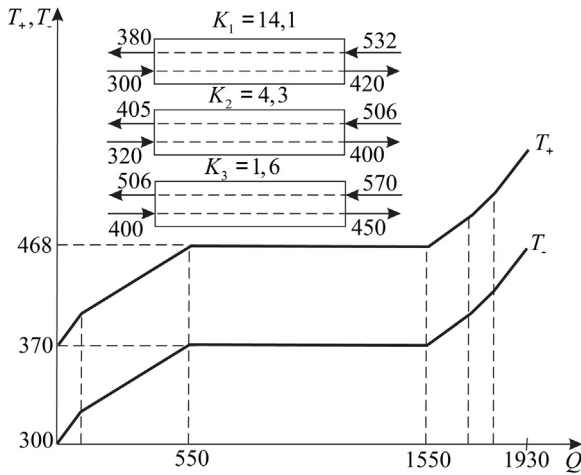
**Fig. 5 – The heat exchange system synthesized in Example and the corresponding temperature profiles $T_+$, $T_-$ in an equivalent dual-stream system. The temperatures (T) are in K, the heat loads (Q) in W, and the heat transfer coefficients (K) in kW/K.**

superheated to a temperature of $T_{1-}^{out} = 420$ K with a heat capacity rate of $W_{1s-} = 2$ kW/K.

The second stream enters and leaves the system in the form of a liquid with a heat capacity rate of $W_{2-} = 5$ kW/K, an inlet temperature of $T_{2-}^{in} = 320$ K, and an outlet temperature of $T_{2-}^{out} = 400$ K.

The third stream enters and leaves the system in the form of a liquid with a heat capacity rate of $W_{3-} = 3$ kW/K, an inlet temperature of $T_{3-}^{in} = 400$ K, and an outlet temperature of $T_{3-}^{out} = 450$ K.

The total heat transfer coefficient for the system is set to $\overline{K} = 20$ kW/K. The physical meaning of this condition is related to the limitation of the total size of the heat exchangers.

Let us show by this example that the conditions of minimum dissipation and equivalence conditions of a dual- and multi-stream system allow us to find the temperature and specific heat of hot streams in contact with cold streams in three dual-stream heat exchangers and the distribution of the heat transfer coefficient $\overline{K}$ between them so that the production of entropy, and therefore the loss of exergy, is minimal.

We calculate the total heat load $\overline{Q} = \overline{Q}_1 + \overline{Q}_2 + \overline{Q}_3 = (70 \cdot 4 + 1000 + 50 \cdot 2) + 80 \cdot 5 + 50 \cdot 3 = 1930$ kW, and we construct the temperature dependence on the heat load $T_-(Q)$ for the equivalent cold stream in the calculated dual-point system. We get

$$T_-(Q) = 300 + \frac{Q}{4} \text{ K at } 0 \le Q \le 80 \text{ W}$$

$$T_-(Q) = 320 + \frac{Q - 80}{(4 + 5)} \text{ K at } 80 \text{ W} \le Q \le 550 \text{ W}$$

$$T_-(Q) = 370 \text{ K at } 550 \text{ W} \le Q \le 1550 \text{ W}$$

$$T_-(Q) = 370 + \frac{Q - 1550}{(2 + 5)} \text{ K at } 1550 \text{ W} \le Q \le 1760 \text{ W}$$

$$T_-(Q) = 400 + \frac{Q - 1760}{2 + 3} \text{ K at } 1760 \text{ W} \le Q \le 1860 \text{ W}$$

$$T_-(Q) = 420 + \frac{Q - 1860}{3} \text{ K at } 1860 \text{ W} \le Q \le 1930 \text{ W}.$$

On each of the sections of the linearity $T_-(Q)$, the integral $I(Q)$ is not difficult to calculate as it was done above for the evaporating stream. So, in the first section, when Q changes from 0 to 80 kW, the integral is equal to

$$I_1 = \int_0^{80} \frac{4dQ}{1200 + Q} = 4 \ln \frac{1280}{1200} = 0.24.$$

As a result of calculating all six integrals and summing them, we get $I(\overline{Q}) = 5.25$ kW/K. Using Eq. (20) we find the optimal ratio of temperatures of equivalent streams and the minimum possible entropy production in the calculated and equivalent multi-stream system as

$$\frac{T_-(Q)}{T_+(Q)} = m = \frac{20}{25.25} = 0.79, \quad \sigma^* = (1 - 0.79) \, 5.25 = 1.1 \text{ kW/K}.$$

The heat capacity rate of the equivalent hot stream $W_+(Q)$ for any heat load must be equal to the heat capacity rate of the cold stream $W_-(Q) = \frac{1}{dT_-/dQ}$ multiplied by $m$. Due to the equivalence condition this requirement also applies to each of the three dual-stream heat exchangers. So, for the first stream, the hot stream temperature at the system inlet is $T_{1+}^{in} = 532$ K, and the heat capacity rate at the inlet is $W_+^{in} = 2 \cdot 0.79 = 1.58$ kW/K.

The heat load of each heat exchanger is equal to the difference between the enthalpy of the cold stream at its outlet and at the inlet. So, for the first of the cold streams, $\overline{Q}_1 = 70 \cdot 4 + 1000 + 50 \cdot 2 = 1380$ kW.

Finally, the coefficient of thermal conductivity which determines the contact surface for each of the heat exchangers is calculated by the formula (35). For Newtonian kinetics, taking into account the fact that the temperature ratio of the contacting streams is constant, will take the form

$$\overline{K}_i = \int_{Q_i^{in}}^{Q_i^{out}} \frac{W_{i-}(Q)dQ}{W_-(Q)T_-(Q)(1/m - 1)}. \tag{36}$$

Here the integral is calculated on the contact interval of the ith stream since outside this interval $W_{i-}(Q)$ is zero.

*The parameters of each of the dual-stream heat exchangers.*

1. The first heat exchanger. For the first heat exchanger into which the first of the cold streams enters, the temperature of the hot stream at the system inlet is $T_{1+}^{in} = 420/0.79 = 532$ K, and its heat capacity rate at the inlet is $W_{1+}^{in} = 2 \cdot 0.79 = 1.58$ kW/K. The condensation temperature of the hot stream is $T_{1+}^b = 370/0.79 = 468$ K, and the flow of transmitted heat, equal to the product of the flow rate of the hot stream on its heat of condensation is 1000 kW. Finally, in the interval of contact with the liquid cold stream, the heat capacity of the hot stream is $4 \cdot 0.79 = 3.16$ kW/K.

   Using the formula (36) for the first heat exchanger, we find the heat transfer coefficient:

$$\overline{K}_1 = \left(\frac{1}{1/0,79 - 1}\right)\left[\int_0^{80} \frac{dQ}{300 + \frac{Q}{4}} + \int_{80}^{550} \frac{4dQ}{9(320 + \frac{Q-80}{9})} + \right.$$

$$+ \int_{550}^{1550} \frac{dQ}{370} + \int_{1550}^{1760} \frac{2dQ}{7(370 + \frac{Q-1550}{7})}$$

$$\left. + \int_{1760}^{1860} \frac{2dQ}{5(400 + \frac{Q-1760}{5})}\right] = 14.1 \text{ kW/K}.$$

Note that the ratio of the specific heat $W_{1-}$ of the first of the cold streams to the total specific heat of the equivalent stream $W_-$ is equal to unity in two sections: the initial one, when the equivalent stream consists of only the first stream, and the evaporation section, when the specific heat of the stream is arbitrarily large.

2. The second heat exchanger. In the second heat exchanger, the hot stream has an inlet temperature of $T_{2+}^{in} = 400/0.79 = 506\,K$, and its heat capacity rate is constant and equal to $W_{2+} = 5 \cdot 0.79 = 3.95\,kW/K$. The heat transfer coefficient for this heat exchanger according to the formula (36) is equal to:

$$\overline{K}_2 = \left(\frac{1}{1/0,79-1}\right)\left[\int_{80}^{550} \frac{5dQ}{9(320 + \frac{Q-80}{9})}\right.$$
$$\left. + \int_{1550}^{1760} \frac{5dQ}{7(370 + \frac{Q-1550}{7})}\right] = 4.29\,kW/K.$$

3. The third heat exchanger. For optimal heating of the third cold stream, the hot stream should have a heat capacity rate of $W_{3+} = 3 \cdot 0.79 = 2.37\,kW/K$, and the inlet temperature $T_{3+}^{in} = 450/0.79 = 570\,K$. The heat transfer coefficient of the third heat exchanger according to the formula (36) is equal to:

$$\overline{K}_3 = \overline{K} - \overline{K}_1 - \overline{K}_2 = 1.61\,kW/K.$$

The temperature dependencies of equivalent streams on the heat load for the system calculated in this example have the same form as in Fig. 4.

The calculated system can be simplified in one way or another, but it provides a guideline to which we must strive, and allows us to evaluate the perfection of the simplified system.

### 4.3. Some hot and cold streams fixed

Let us consider a general case when the parameters of several hot and several cold streams are fixed. Like previously, the total heat transfer coefficient of the system, $\overline{K}$, is specified. However, it is necessary to divide the system into two subsystems, one for which only cold stream parameters are set, and another for which only the hot streams are specified. The synthesis of such systems has already been considered. What is new is that it is necessary to optimally allocate $\overline{K}$ between these subsystems.

Let us denote the values related to a system with fixed hot streams by the index "+" and those with fixed cold streams by the index "−". The heat loads of each of the subsystems $\overline{Q}_+$ and $\overline{Q}_-$ are set via the heat capacity rates and input and output temperatures of the fixed streams. The entropy production is equal to the total entropy production of all subsystems, and the total heat transfer coefficient $\overline{K}$ must subsequently be distributed among them so that the total entropy production in the system is minimal. In mathematical form the problem is

$$\sigma^* = \sigma_+^*(\overline{K}_+, \overline{Q}_+) + \sigma_-^*(\overline{K}_-, \overline{Q}_-) \longrightarrow \min_{\overline{K}_+, \overline{K}_-} \tag{37}$$

under the condition

$$\overline{K}_+ + \overline{K}_- = \overline{K}. \tag{38}$$

The Lagrange function of the problem will take the form

$$L = \sigma_+^*(\overline{K}_+, \overline{Q}_+) + \sigma_-^*(\overline{K}_-, \overline{Q}_-) + \lambda(\overline{K}_+ + \overline{K}_-). \tag{39}$$

The stationarity conditions of this function for the desired variables, after eliminating the Lagrange multipliers, lead to the equation

$$\frac{\partial \sigma_+^*}{\partial \overline{K}_+} = \frac{\partial \sigma_-^*}{\partial \overline{K}_-} \tag{40}$$

which says that the incremental entropy productions must balance when the heat conduction split is optimal. Together with the condition (38) this determines the desired distribution of heat transfer coefficients between subsystems.

For Newtonian kinetics the condition (40) becomes

$$\left(\frac{1 - m_+}{m_+}\right)^2 = \left(\frac{1 - m_-}{m_-}\right)^2. \tag{41}$$

Since both sides of (41) depend monotonically on their arguments, we obtain the general rule: *The coefficients of thermal conductivity between the two parts of the heat exchange system must be distributed such that the absolute temperature ratio of the contacting streams are the same for the entire system.* This simple rule ties all the components of the heat exchange system together in precisely the balance which ensures overall optimality (Salamon et al., 1980).

Let us write the equation that follows from this condition and the equalities (18) and (21):

$$m_- = \frac{\overline{K}_-}{I(\overline{Q}_-) + \overline{K}_-} = m_+ = 1 - \frac{J(\overline{Q}_+)}{\overline{K}_+}, \tag{42}$$

where

$$I(\overline{Q}_-) = \int_0^{\overline{Q}_-} \frac{dQ}{T_-(Q)}, \quad J(\overline{Q}_+) = \int_0^{\overline{Q}_+} \frac{dQ}{T_+(Q)}, \tag{43}$$

and $T_-(Q)$, $T_+(Q)$ are temperatures of equivalent cold and hot streams which combine the fixed streams of each of the subsystems. After eliminating $\overline{K}_+ = \overline{K} - \overline{K}_-$ in the condition (42), we get an expression defining the optimal distribution of the total thermal conductivity coefficient between subsystems:

$$\overline{K}_- = \frac{I(\overline{Q}_-)(\overline{K} - J(\overline{Q}_+))}{J(\overline{Q}_+) + I(\overline{Q}_-)}, \quad \overline{K}_+ = \frac{J(\overline{Q}_+)(\overline{K} + I(\overline{Q}_-))}{J(\overline{Q}_+) + I(\overline{Q}_-)}. \tag{44}$$

Substituting these values back into Eq. (42), we can determine the temperature ratio of the contacting streams for the system, $m_+ = m_- = m$. With this distribution the minimum entropy production in the system is equal to the sum of entropy production in subsystems, each of which is expressed in terms of the value found for $m$ in accordance with (20) and (22).

*Boundary of physical feasibility of heat exchange systems.* For an arbitrary designed heat exchange system in which the heat transfer coefficient, heat load, and laws of temperature change of hot or cold streams during their cooling or heating are known, the minimum possible entropy production $\sigma^*$ can be found using the above procedure and formulae. The actual entropy production is

$$\sigma = \sum_j W_j \ln \frac{T_j^{out}}{T_j^{in}},$$

where summation is over all streams, both hot and cold. Comparing $\sigma$ with $\sigma^*$ shows the degree of thermodynamic perfection of the designed system. The ratio

$$\sigma^*/\sigma \leq 1 \tag{45}$$

is the efficiency of the heat exchange complex, equivalent to the 2nd law efficiency of a heat engine. The equality part of this inequality highlights the boundary of realizable systems in the space of the output temperatures as expressed in terms of parameters such as heat transfer coefficients and heat capacity rates of streams.

## 5. Conclusion

General conditions for minimal dissipation of the heat exchange process are given in Eqs. (9) and (10) or alternatively (11). They are elaborated for different forms of heat transfer dependencies, e.g. Newton, Fourier, and radiative heat transfer, in Sections 3.1 and 3.2.

The concept of thermodynamic equivalence of dual-stream and multi-stream systems is introduced and the conditions that lead to such equivalence are obtained in Section 4.1. Using this equivalence for Newtonian heat transfer, optimal synthesis conditions are obtained for dual-stream and multi-stream systems with a given total heat transfer coefficient and heat load. Further, a lower bound for entropy production is derived, corresponding to the minimum possible exergy loss in the heat exchange process. In most cases the temperature profile of some hot and some cold streams are set and the characteristics of the other streams are subject to optimization.

A method is proposed for calculating the parameters of countercurrent heat exchangers (heat capacity rates, stream temperatures, thermal conductivity coefficients) in a multi-stream system which will indeed reach the lower limit of irreversibility. The number of heat exchangers is equal to the number of streams with fixed parameters. Such a synthesized system can serve as a reference when evaluating the thermodynamic perfection of an existing system simply by comparing the entropy production of the actual system with that of the optimized system. In other words, the lower bound of irreversibility determines the region of realizability of heat exchange systems.

## Declaration of Competing Interest

The authors report no declarations of interest.

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