Research Article

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The Ideal Gas in Slow Time

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Abstract: We continue our exploration of thermodynamics at long observational timescales, "slow time," by including turbulent dynamics leading to a condition of fluctuating local equilibrium. Averaging these fluctuations in wind speed and temperature results in a velocity distribution with heavy tails which, however, are necessarily truncated at some large molecular speed preserving all moments of the velocity distribution including the energy. This leads to an expression for the ideal gas law in slow time which as its core has the superficially familiar term $\frac{3}{2}Nk\theta$ in addition to a term accounting for the large-scale fluctuations, which is also proportional to the particle number N; θ is a new temperature including thermalization of wind. The traditional temperature T no longer exists. Likewise, the additional energy term necessitates a new quantity that parallels entropy in the sense that it captures hidden degrees of freedom. Like entropy, it captures physical properties manifesting indirectly, but on scales larger than the familiar laboratory scales. We call this quantity epitropy.

Keywords: slow time, long time, ideal gas law, epitropy

1 Introduction

Local equilibrium is a ubiquitous condition in physical systems, from planetary atmospheres to oceans and stars. It is of necessity in the context in which thermodynamics was first envisaged and in which it is commonly experienced. These systems need not be in steady states. Local equilibrium typically coexists with turbulent dynamics leading to a condition of *fluctuating local equilibrium* (FLE). This paper aims to examine FLE systems on long timescales using what we have developed previously [1] within the conceptualization of *slow time*. For another interesting approach to fluctuating systems see superstatistics introduced by Beck [2].

The slow-time project aims to treat directly the effects of long timescales on physics or chemistry as time resolution is lost. The distinct regimes of physics, such as the atomic and laboratory regimes, are familiar. These of course are special cases of quantum mechanical and relativistic regimes. There is a clear hierarchy which has the distinctive property that each regime can "ignore" underlying ones even if they must be in fundamental agreement. That property is best known as *closure* [3], following the terminology arising from the historical problem of turbulence. Closure emerges through a process that induces new relationships between existing or modified quantities in some limit, yielding a system of equations that can be solved without reference to the underlying regime.

The central question the slow-time approach asks is whether a new regime or new regimes emerge on long timescales and correspondingly coarsened space scales. Loosely speaking that question puts an observer in a situation not unlike trying to view the laboratory regime from atomic or kinetic scales. From the standpoint of such scales the laboratory regime induces new physical variables such as temperature, while burying specific dynamical variables in the loss of resolution whereby entropy emerges.

In the prior work [1], free variables (temperature and flow velocity) were fluctuating in a simple homogenous system to produce probability density functions (PDFs) appropriate for long timescales. It attempted to

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identify key features of PDFs that must arise without consideration of local equilibrium systems per se. FLE systems go one step further and introduce new considerations as they are embedded in a configuration space. Instead of a single fluctuating thermodynamical-dynamical system, FLE implies the existence of fluctuating scalar and vector fields throughout the system, i. e., in this view the large system does not fluctuate in unison but each local pocket may fluctuate by itself, hence the need for fields.

This generalization to FLE systems has mostly indirect effects on the reasoning of the prior work. Thus we do not directly repeat the same analysis in 3D that we performed in 1D earlier. Instead we operate with the ansatz that the key features of the PDF approximation derived there,

$$p_{uT}(v,w) = \frac{\psi_0}{\sqrt{\pi}}\left(\frac{w^2}{v^2+w^2}\right)^{3/2} e^{-\frac{w^2\psi_0^2v^2}{v^2+w^2}}, \tag{1}$$

persist. In this equation v is the particle velocity and $\psi(\theta)$ is the precision of the Gaussian velocity distribution which is itself fluctuating around the central value

$$\psi_0 \equiv \psi(\theta)\Big|_{\theta=\theta_0} = \sqrt{m/(2k\theta_0)}. \tag{2}$$

Here θ is the *thermalized wind temperature*, which is a generalization of temperature that includes thermalized wind [1]; w is the Gaussian precision of the fluctuations in ψ with units of velocity. For mathematical convenience we work with the Gaussian precision instead of the standard deviation. For standard deviation σ the precision $\psi = 1/\sigma$. Thus a larger precision means a tighter distribution. The subscripts of p_{uT} indicate fluctuated velocity, u, and temperature, T. (In [1] p_{uT} was denoted $\hat{p}(v;w,\psi_0)$.)

Equation (1) is a simplified, but highly accurate, approximation of the full PDF from [1]. The simplified version may be as sound as the full version in that the full version is based on certain ad hoc assumptions such as Gaussian fluctuations, and in the case of temperature a truncated Gaussian. While these are plausible they are not confirmed. Any highly accurate simplification may be just as valid. This is in line with our goal at this stage to illustrate how different slow time is from behavior at the laboratory scale. A full general thermodynamic theory at slow time is still to come.

The PDF normalization is not fully maintained in this approximation as it can be shown that $p_{uT}(v)$ normalizes to $\mathrm{erf}(w\psi_0)$, which is very close to 1 provided $w\psi_0 \gg 1$. The physical interpretation of this requirement is that the standard deviation $(1/w)$ of the fluctuations ψ of the velocity fluctuations must be suitably smaller than its central value ψ_0 [eq. (2)], such that most of the weight of the distribution $\psi(\theta)$ remains within the physical regime of $\theta \geq 0$. If that is not the case, the distribution is no longer Gaussian as assumed but has substantial weight in the unphysical range of negative temperatures.

Numerical integration shows that in practice only $w\psi_0 \gtrsim 2$ is needed [1]. The conditions discussed in [1] estimated values of $w\psi_0$ larger than 2 by at least a factor of 10^2. For perspective, $\mathrm{erf}(10^2) \approx 1 - 10^{-4345}$. Thus the probability lost in the simplification is extremely small, justifying such an approximation for a PDF.

The term w is the *transition velocity* at which the effects of temperature fluctuations manifest as being unthermalizable. This can be seen in the factor $\exp[-\frac{w^2\psi_0^2v^2}{v^2+w^2}]$, discovered in the prior calculations, which has extraordinary properties that complement the more familiar properties of the remaining factors. The transition velocity marks where this factor switches from Gaussian for small v to a constant for large v. Similarly the transition velocity marks where the factor $\frac{\psi_0}{\sqrt{\pi}}\left(\frac{w^2}{v^2+w^2}\right)^{3/2}$ changes from a constant for small v to a power law for large v, leading to a heavy tail PDF [1]. The result is a Gaussian core with heavy tails.

This paper develops from those previous results [1]. The properties employed for the FLE case are in three categories: previous results, fundamental requirements, and simplifying assumptions.

Previous results [1]

- PDFs are a composite of Maxwellian behavior for small velocities and heavy tails for large velocities.
- There is a sharp transition between the two behaviors at the critical transition velocity, w, with a full Gaussian form returning in the large w limit.

– Heavy tails imply that for finite w the distribution is not Gaussian and hence temperature is a laboratory regime property that does not exist in the slow-time limit. This arises because, unlike persistent velocity ("wind") fluctuations which are absorbed into the new temperature θ [1], temperature fluctuations cannot be thermalized. "Thermalization" means that Gaussian fluctuations are incorporated only into changes in the standard deviation of a Gaussian form.

Fundamental requirements

– FLE systems reduce to thermodynamic equilibrium in the absence of fluctuations. For example, in (1) as $w \to \infty$, the distribution of fluctuations in temperature goes to a delta function and (1) becomes Gaussian. With this principle the slow-time thermodynamics excludes distributions that do not reduce to standard thermodynamics at laboratory-scale conditions. This condition plays a role similar to the correspondence principle of quantum mechanics, or the Newtonian limit of relativity.
– Heavy tail behavior holds for finite velocity magnitudes greater than w, but is asymptotically artificial, meaning that moments must exist in true distributions because these distributions become depopulated beyond large enough $v = v_*$, where it is presumed that $v_* \gg w$. If instead $v_* \ll w$, one can proceed with a truncated Gaussian form without heavy tails. We will not consider that here, but it should be straightforward to implement using these results.

Simplifying assumptions

– The velocity distribution is isotropic allowing meaningful representation by a single velocity axis. Anisotropy can be accounted for if needed elsewhere.
– Fluctuating mean velocity ("wind") is assumed fully thermalized for simplicity, according to [1]. However, persistent non-fluctuating winds can be easily accounted for too if desired.

This paper uses these properties to determine a slow-time relationship between moments in an FLE system that does not employ usual variables in the laboratory regime valid for finite w. It builds on knowledge of Gaussian systems and introduces a deviation function R, known as the redistribution function, to account for non-Gaussian behavior arriving finally at a long timescale equation for an ideal gas in an FLE system.

2 Isotropy conditions and normalization

2.1 One dimension

In [1] we focused on developing a PDF strictly in one space dimension. In order to make this physically meaningful, we must discuss how this pertains to a 3D velocity space. For simplicity we will aim for isotropic probability distributions as is often the case in thermodynamics.

We have already discussed how extraordinarily small the deviation from 1 is in the normalization of the approximate PDF (1). If one seeks a regular normalization, one may, most naturally, regularize the lost probability from (1) by adding a minute term that integrates over v to give $1 - \operatorname{erf}(w\psi_0)$. Such a function is not unique, which merely reflects the highly accurate though approximate nature of (1) as long as $w\psi_0 \gtrsim 2$. If ψ_0 is taken as a given constant for this analysis and thus by this requirement $w \gtrsim 2/\psi_0$, one could use any function $\beta(v, w)$ with a finite interval $(-\gamma, \gamma)$ and chosen to satisfy all the following properties,

(a) $0 \le \beta(v, w) \ll p_{uT}(v, w)$ for $v \in (-\gamma, \gamma)$,
(b) $\beta(v, w) = 0$ for $v \notin (-\gamma, \gamma)$,
(c) $\int_{-\gamma}^{\gamma} \beta(\tau, w)\, d\tau = 1 - \operatorname{erf}(w\psi_0)$, $w\psi_0 \gtrsim 2$,

to regularize the normalization. There is no difficulty to achieve these with the characteristic numbers in play.

2.2 Three dimensions

We will follow the same method for regularization in 3D as just outlined for 1D above. For those requirements, v and w are 1D quantities in the $\mathbb{R}^1$ velocity space. However extending into $\mathbb{R}^3$ requires a distinction between the values on an axis, i (i. e., v_i, w_i), and the magnitude of vectors (i. e., v, ω). We will employ this distinction in the remainder of this section.

Thus by analogy we now introduce the small 3D regularization term ξ through

$$p(v,\omega,\alpha_1,\alpha_2,\alpha_3)\ dV = \xi(v,\omega,\alpha_1,\alpha_2,\alpha_3)dV + \frac{\psi_0^3}{\pi^{3/2}}\left(\frac{\omega^2}{v^2+\omega^2}\right)^{9/2} e^{-\frac{\omega^2\psi_0^2v^2}{v^2+\omega^2}}\,dV, \tag{3}$$

where the three components of the regularized probability, $\beta(v\alpha_i,\omega\alpha_i)+p_{uT}(v\alpha_i,\omega\alpha_i)$, were multiplied. The α_i are the direction cosines (i. e., $\alpha_1^2+\alpha_2^2+\alpha_3^2=1$ and $v_i=v\alpha_i$, $w_i=\omega\alpha_i$) for each of the three components in the 3D space.

It is tempting to regard ξ as an inherited regularization from the 1D cases. However, with such an interpretation complications arise because of conditions that are excluded by the 1D regularization properties. It should be kept in mind that the volume element dV in (3) is what must be infinitesimal, not necessarily its extension in each of the Cartesian directions. Moreover, attempts to manage them involve 1D structures that do not manifest in 3D. In the first instance, ξ is the sum of products of individual terms; β never appears alone. By far most of the domain will satisfactorily follow the 1D regularization requirements, but they will not be valid for $\alpha_i \lesssim 2/(\omega\psi_0)$, which corresponds in special cases to angles about a degree off a critical axis. This is a domain excluded by the 1D regularization conditions. While one might try to relax this condition by careful choices of β, it cannot eliminate the singular case at $\alpha = 0$. There $w = 0$, which is non-physical and mathematically problematic (the limit does not exist at the origin).

These difficulties arise entirely because of trying to insert 3D considerations into a perfectly sufficient 1D regularization. Moreover they are entirely a spurious artifact of the choice of coordinate system. That is, if the actual manifold is left fixed and the coordinate system is rotated, the difficulties follow the axes and do not remain with the manifold. This problem is manifested if one considers the integrals of (3),

$$\int_D \xi(v,\omega,\alpha_1,\alpha_2,\alpha_3)dV = 1 - \int_D \frac{\psi_0^3}{\pi^{3/2}}\left(\frac{\omega^2}{v^2+\omega^2}\right)^{9/2} e^{-\frac{\omega^2\psi_0^2v^2}{v^2+\omega^2}}\,dV, \tag{4}$$

where D represents the entire space. The right side of (4) does not depend on $\{\alpha_i\}$, while the left side apparently does. But it does not need to. The only connection ξ has to reality is its role in regularizing p. Without loss we may require ξ to be some new function such that $\xi(v,\omega,\alpha_1,\alpha_2,\alpha_3) \to \xi(v,\omega)$, which solves this issue as a sufficient but not necessary requirement:

$$\int_D \xi(v,\omega)dV = 1 - \int_D \frac{\psi_0^3}{\pi^{3/2}}\left(\frac{\omega^2}{v^2+\omega^2}\right)^{9/2} e^{-\frac{\omega^2\psi_0^2v^2}{v^2+\omega^2}}\,dV. \tag{5}$$

Equation (5) is the correct 3D analog to the requirement (c) of the 1D regularization properties. So long as $\omega\psi_0$ follows the characteristic values of $w\psi_0$, the other analogs to the 1D properties will follow. Thus

$$p(v,\omega)\ dV = \frac{\psi_0^3}{\pi^{3/2}}\left(\frac{\omega^2}{v^2+\omega^2}\right)^{9/2} e^{-\frac{\omega^2\psi_0^2v^2}{v^2+\omega^2}}\,dV, \tag{6}$$

where ξ has been discarded due to its minuteness and the same replacement $p(v,\omega,\alpha_1,\alpha_2,\alpha_3) \to p(v,\omega)$ as above for ξ has been applied, leaving an isotropic expression for $p(v,\omega)$. Of course ξ need not be isotropic. It was only chosen to be. If one repeats this argument for the $w \to \infty$ case, this entire argument goes over at each step to the classical development of a 3D distribution for the Maxwellian.

Isotropy is of course unnecessary. Anisotropy can be accommodated building from the 1D case too. Anisotropy conditions are commonplace in physics, for example, supersonic molecular beams which have

different temperatures by a large factor in different directions. (See for example Section 2 of [4], Section 3.4 of [5], or [6].) Similarly major directional differences are found in jets emerging from neutron stars and black holes due to magnetic fields. Similarly photons have states embedded in momentum space which induce inherently direction-dependent states [7]. Only in full thermodynamic equilibrium do all of these direction-dependent states coalesce into one, achieving isotropy. What is achieved in this section is that we can mimic the approach of Maxwell to make the 1D approach that we have developed have an impact in a normal 3D space. Rearranging probability has a more significant role in what follows.

In the following, unless stated otherwise, we suppress the subscript i and let v represent one of the components v_i, and similarly w will represent w_i.

3 The slow-time probability density function

In [1] new PDFs were deduced from the Maxwellian, $p_M(v)$ [previously $p(v,T)$], by fluctuating that distribution's parameters. When flow velocity was fluctuated about zero, the wind was "thermalized," yielding another Gaussian form,

$$p_u(v) = \frac{\psi}{\sqrt{\pi}}\, e^{-\psi^2 v^2} \tag{7}$$

[previously $p(v,\theta)$], where ψ is defined in (2) above. This yields three PDFs, p_M, p_u, and p_{uT}, to which we add a fourth, $p_s(v)$, which is the true slow-time PDF.

The PDF p_s not only encompasses the loss of spatial resolution that slow time implies, but it also gathers the full range of velocity distributions of the field into a single distribution, which we presume is isotropic according to the description in the introduction. In the following we will thus address this distribution by considering only one axis unless otherwise specified.

While p_{uT}, as per (1), asymptotically approaches zero, p_s must drop to zero abruptly after some velocity v_* because the real world is not a continuum but consists of a finite number of particles, i. e., the wing of the distribution is granular with one particle being the fastest. If the entire integration domain, D, is $D = D_1 \cup D_2$ where $D_1 = \{v | v_* \geq |v|\}$ and $D_2 = \{v | v_* < |v|\}$, then

$$p_s = \begin{cases} p_{uT} + \delta(v) & v \in D_1, \\ 0 & v \in D_2, \end{cases} \tag{8}$$

where $\delta(v)$ is a correction function that ensures that normalization is preserved. A constraint on δ follows,

$$\int_{D_1} \delta dv = \int_{D_2} p_{uT} dv = 2\int_{v_*}^{\infty} \frac{w^3\psi_0}{\sqrt{\pi}(v^2+w^2)^{3/2}}\, e^{-\frac{w^2\psi_0^2 v^2}{v^2+w^2}}\, dv. \tag{9}$$

For small $\zeta = w/v$,

$$\frac{w^3\psi_0}{\sqrt{\pi}(v^2+w^2)^{3/2}}\, e^{-\frac{w^2\psi_0^2 v^2}{v^2+w^2}} = \frac{\zeta^3\psi_0}{\sqrt{\pi}} e^{-w^2\psi_0^2}(1+O(\zeta^2)). \tag{10}$$

Thus for large $v = v_*$,

$$\int_{D_1} \delta dv \sim 2\frac{w^3\psi_0}{\sqrt{\pi}} e^{-w^2\psi_0^2} v_*^{-2} = 2\frac{w\psi_0}{\sqrt{\pi}} e^{-w^2\psi_0^2}\zeta_*^2. \tag{11}$$

Representative values from the appendix of [1] put values for the exponent of the exponential factor $\sim -10^4$ (i. e., 10^{-4342} from the introduction). Thus the correction is exceptionally small.

We may draw two conclusions if (8) holds:

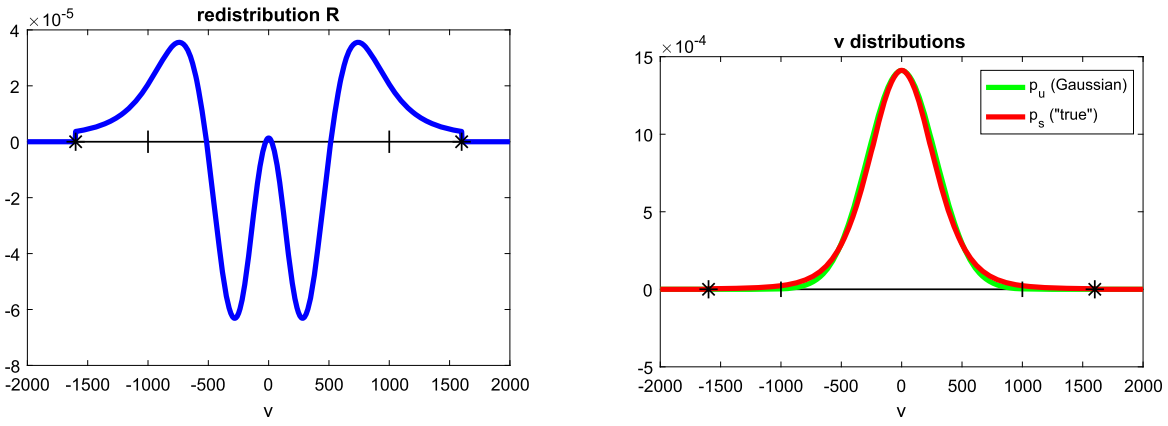

Figure 1: Distribution functions for $w = 1000$ (marked with a | on the v axis) and $v_* = 1600$ (marked with a *). The left frame shows the redistribution function $R(v) = p_s - p_u$ which is equal to $-p_u$, typically an exceptionally small quantity, for $|v| > v_*$. The right frame shows its two components, the Gaussian p_u and the "true" distribution p_s which vanishes identically for large $|v|$.

1. Setting $\delta = 0$ in (8) is an exceptionally accurate approximation. This is of importance when using p_{uT} to estimate p_s when $v < v_*$.
2. All moments of p_s, the true distribution (8), converge because $p_s = 0$ for $v > v_*$. This conclusion is not a direct consequence of the previous one, but primarily based on the existence of an upper bound on speed, v_*.

4 The redistribution function *R*

While the list of properties laid out in the introduction call for the true distribution function p_s to be non-Gaussian, our classical conceptualization of such problems is Gaussian in form. A Gaussian PDF is a natural reference, thus we define the redistribution function

$$R(v) \equiv p_s - p_u. \tag{12}$$

The integral of R over the full domain D must be

$$\int_D R\, dv = 0, \tag{13}$$

because probability is conserved. Since R is an even function it follows that

$$\int_0^\infty R\, dv = 0. \tag{14}$$

This means that excesses in the differences of probability must cancel out. Deficits in one domain must cancel with excesses in another. We know that heavy tails are larger than exponential tails, so for large v, R must be positive. This must be compensated for with a domain of negative values. Thus there must be at least one zero in R. Moreover, near $v = 0$ the function is Maxwellian, so there must be a second zero and horizontal tangent there. See Fig. 1. The positive and negative areas of R are indeed equal as required by (14). This shape agrees with the function $\Phi = p_{uT}/p_u$ (Fig. 2) discussed in [1], which is even with two local minima and a local maximum at $v = 0$; R sets p_s up as a perturbation on a familiar form: $p_s = p_u + R$.

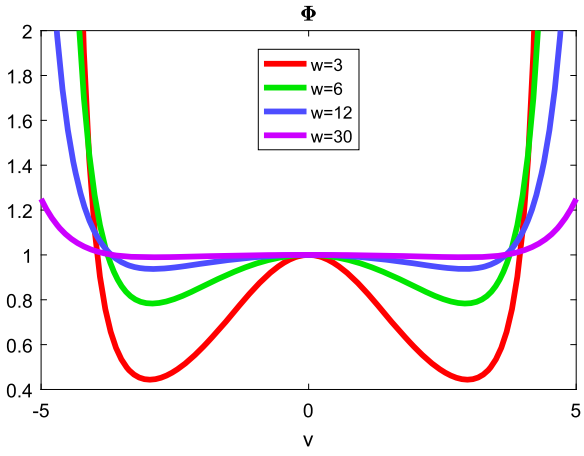

Figure 2: The function $\Phi = p_{uT}/p_u$ relating the PDF for fluctuating velocity as well as temperature for four different precisions of precisions w to the PDF with an exact precision of temperature. The two minima disappear over a fairly small w range. Note that v and w are on different scales from the previous figure in order to show this effect clearly.

5 The slow-time ideal gas law

Section 3 demonstrates that all moments of p_s are convergent. Let $\mathcal{M}_n\{g(v)\} \equiv \int_D v^n g(v)\,dv$ be the nth moment of $g(v)$. Then we may compute $\mathcal{M}_n\{p_s(v)\}$. With the assumptions from the introduction, we find that $\mathcal{M}_0\{p_s(v)\} = 1$ and $\mathcal{M}_1\{p_s(v)\} = 0$.

Second moments are most relevant in connection with energy, E, which involves all components of v,

$$E = \frac{mN}{2}\sum_i \mathcal{M}_2\{p_s(v_i)\} = \frac{mN}{2}\sum_i \mathcal{M}_2\{p_u(v_i)\} + \frac{mN}{2}\sum_i \mathcal{M}_2\{R(v_i)\}, \tag{15}$$

where N is the number of particles in the entire FLE system. The form of the second sum, involving p_u, follows naturally from known integrals, so

$$E = \frac{3}{2}Nk\theta + \frac{mN}{2}\sum_i \mathcal{M}_2\{R(v_i)\}. \tag{16}$$

While from properties of R, $\mathcal{M}_0\{R(v)\} = 0$ and $\mathcal{M}_1\{R(v)\} = 0$, we cannot expect the same result for $\mathcal{M}_2\{R(v)\}$. In [1] we found $\mathcal{M}_2\{p_{uT}(v)\}$ to diverge logarithmically even if $\mathcal{M}_2\{p_s(v)\}$ does not. For some suitable function $\delta(v)$ in (8), then

$$E = \frac{3}{2}Nk\theta + Nf(w, v_*), \tag{17}$$

where the properties of $f(w, v_*)$ require w, v_*, and δ to be fully determined in terms of the specific structures in play.

The correspondence principle for the FLE systems requires that the energy equation reduces to the classical equation $E = \frac{3}{2}NkT$ in the absence of fluctuations. Clearly $\theta \to T$ in that case, but we must also have $\lim_{w\to\infty} R(v) = \lim_{w\to\infty} f(w, v_*) = 0$. This follows if $\lim_{w\to\infty} p_s(v) = p_u$, which in turn implies $\lim_{w\to\infty} \delta(v) = 0$. To ensure these results we must either ignore v_* or avoid depopulated domains by requiring $w \ll v_*$. If we assume to the contrary that v_* is held fixed, then the redistribution function and the normalization correction do not vanish in the limit, thus the moments do not generally vanish in that limit (i. e., $\lim_{w\to\infty} f(w, v_*) \neq 0$), and the correspondence principle does not apply. This allows us to consider an expansion at infinity of the form $f(w, v_*) = w^{-2}h + O(w^{-4})$, or

$$E = \frac{3}{2}Nk\theta + Nw^{-2}h + O(w^{-4}), \tag{18}$$

where h is a constant to be determined. Outside of possible exceptions we do not at present know this slow-time behavior and thus must expect $h \neq 0$. Its value will depend on specifics unavailable to us *a priori*, but it is in principle something measurable with commensurate time and instruments.

Discarding higher order terms in (18),

$$E = \frac{3}{2}Nk\theta + Nw^{-2}h \tag{19}$$

becomes the slow-time version of the ideal gas law, where θ, w, and h are the natural regime measurables.

6 Discussion and conclusion

To summarize, in slow time θ reduces to T in the no-fluctuation limit, but θ must not be confused with T, as it includes "thermalized" wind. Classical temperature T no longer exists. The transition velocity, w, is the statistical precision of fluctuations in the temperature variable, ψ [eq. (2)]. It represents the transition from Gaussian to polynomial behavior. The term h is the residual at infinity of the correction term in the slow-time ideal gas law (19).

Extensive and intensive variables

Classical thermodynamics makes a sharp distinction between extensive and intensive variables; see, e. g., [8]. Extensive variables scale linearly with the size of the system (e. g., particle number, energy, and volume, to name the most common set). Intensive variables are the partial derivative of some objective functions with respect to these extensive variables (e. g., chemical potential, temperature, and pressure, corresponding to the above). They are independent of the size of the system. Thus extensive and intensive variables always come in pairs. The common objective functions (potentials) are $E(S, V, N)$ and $S(E, V, N)$. They both operate within the same set of extensive variables, $\{E, S, V, N\}$, but with different intensive variables.

In slow time we will talk about variables that scale linearly with the particle number N as extensive, e. g., the energy E in (19). However, the corresponding intensive variables are more delicate. If we choose E as our objective function, its derivative with respect to N would be

$$\chi = \frac{\partial E}{\partial N} = \frac{3}{2}k\theta + w^{-2}h. \tag{20}$$

The notion of volume, which could have given us a pressure, was treated implicitly. That is, in part, because, along with coarsening time resolution, space is also coarsened. That could either be interpreted as volume being unbounded, in which case there is no pressure, or we could work in the local equilibrium approximation and define volume as the region of space occupied by a certain number of particles. This is typically what is done when calculating the pressure at different altitudes of the atmosphere. The precise form in slow time is something for future work.

Hoping to define an appropriate slow-time temperature, distinct from θ, analogously to the laboratory regime requires that we have a definite entropy of the FLE in order to be able to calculate a derivative analogous to $\partial E/\partial S$. However, there is no fundamental reason to carry over the notion of intensities generated from partial derivatives of a function of extensities. Like classical temperature, this analogy may prove in future work to be unsuitable for the slow-time regime.

Epitropy

Entropy does not exist at the atomic scale since all motion in principle can be monitored and thus is represented as kinetic energy. On the laboratory scale that small random motion is coarse grained away (blurred) but its average effect remains in the form of entropy. In slow time an analogous effect turns randomly variable winds into a temperature contribution to the new thermalized wind temperature θ [1].

However, h is different in that it represents the value of integrals over a domain that is truncated by depopulation in a manner external to classical theories, which assume an inappropriate infinite or semi-finite domain with non-zero integrands. The integral h should have a measurable value, but that value is not knowable from classical theoretical principles. The theoretical determination of h depends on properties that we cannot access at present but nonetheless must collectively manifest themselves.

It seems on its face to be paradoxical: something integral to the machinery that may be observed while depending on things inaccessible to us. But we deal with such a quantity in the form of entropy. The etymological meaning of entropy might be best translated for this purpose as internal churning. We do not see the activity because it is internal. In the case of h we do not see its hidden activity because it is external, so the etymological analog is *epitropy*, from Greek *epi* ("external") and *trope* ("churning"). The new term in the slow-time ideal gas law, h, captures features that are invisible on shorter timescales, e. g., the laboratory scale, which accrue to emerge in the slow-time regime. In principle h is a new observable of the slow-time regime. However more properties of it will be explored in future work.

Isotropy

The assumption of isotropy is not necessary. One can imagine actions under fields that set preferential directions, for example.

Persistent winds

The assumption that the flow velocity, $\mathbf{u}$, averages to zero is unnecessary. One can imagine a wind field that is fluctuating plus a persistent component. This will cause the first moment among others to no longer be zero, and will lead to dynamical equations, distinct from thermodynamical ones. That is the topic of future work.

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