Convective heat transfer law for an endoreversible engine

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(Received 6 October 2005; accepted 5 May 2006; published online 12 July 2006)

A generic model of an endoreversible engine is developed for studying the effect of convective heat transfer, the rate of which depends on the temperature difference to the power n where n is close to unity. The efficiency at maximum power production is found to have as its principal part the Curzon-Ahlborn [Am. J. Phys. **43**, 22 (1975)] expression and a small correction which depends slightly on the temperature ratio of the heat engine reservoirs and the relative heat conductances to the hot and cold sides. By a proper choice of the independent variables it is demonstrated that the analysis becomes simple and approximate analytical expressions are easily derived. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2212271]

I. INTRODUCTION

Finite-time thermodynamics extends thermodynamic analysis to include finite-time constraints, e.g., nonvanishing heat transfer rates, friction, and heat leaks.[1] The effect of the heat transfer mode for endoreversible heat engines was studied, assuming a temperature dependence to the power n, $q \propto T_1^n - T_2^n$ ($n=1$ for Newtonian heat transfer and $n=4$ for radiative heat transfer),[2–4] and temperature difference to the power n,[8] $q \propto (T_1 - T_2)^n$. Analysis of the endoreversible heat engine with Newtonian heat transfer law led to the conclusion that the efficiency at maximum power production is independent of the heat conductance,[5] the so called Curzon-Ahlborn efficiency,

$$\eta_{CA} = 1 - \sqrt{T_C/T_H}, \tag{1}$$

where T_C and T_H are the cold and the hot reservoir temperatures between which the endoreversible heat engine works. This conclusion is found to be no longer valid when a nonlinear heat transfer law is used.[2,4] For example, in Ref. 6 the temperature difference dependence using the power $n=5/4$ has been considered.

The Curzon-Ahlborn efficiency is remarkable in that it, like the Carnot efficiency, depends only on the ratio of the reservoir temperatures.[5] Furthermore, this expression was found to agree quite well with observed efficiencies of real power plants.[5] However, this agreement was discussed and shown to be fortuitous.[7]

The power n of the temperature difference for convection heat transfer was assumed to be $n=5/4$ and later modified to fit the observed efficiency of heat engines.[6] However, there are many other involved sources of losses to affect the empirical value of the efficiency of heat engines or power plants, e.g., heat leaks, fluid friction, transformer and generator losses, power to run pumps and compressors, and dc/ac conversion efficiency, not just those of the thermal cycle itself. Thus the measured efficiencies are bound to be less than the theoretical ones and there is no physical basis to modify the value of n in order to better match the empirical data.

The chosen value of $n=5/4$ is based on natural convection heat transfer analysis.[8] For this specific choice of n the efficiency of the endoreversible engine was derived numerically[6] and the efficiency found by solving two coupled nonlinear equations simultaneously.

In the present study we will consider the temperature difference dependence to the power n where n is close to unity, $n=1+\varepsilon$, and ε is small. This type of dependence is often used in convective heat transfer analysis, and the power n is close to unity.[8] The efficiency at maximum power w is derived as an approximate analytic expression which is shown to have a principal part, η_{CA}, and small correction, i.e., $\eta_w = \eta_{CA} + \eta_1 \varepsilon$, where η_1 depends on both the heat reservoir temperature ratio and the heat conductance ratio κ for the couplings to the two heat reservoirs. In the current analysis we choose the time division s (the fraction of the total cycle time during which the working fluid is coupled to the hot reservoir) and the thermal efficiency of the engine as the two independent variables involved in the problem. It will be shown that this choice of variables simplifies the analysis. The results derived in the present study are different from those derived earlier[6] since we perform the analysis in general terms and derive analytic equations for the efficiency at maximum power output.

In Sec. II the model is introduced and the solution is given in Sec. III. Finally, Sec. IV presents the discussion and our conclusions.

II. MODEL

In dealing with convective heat transfer processes the following heat transfer law is often used:

$$\dot{q} = \frac{dQ}{dt} = \kappa_s (\Delta T)^n, \tag{2}$$

where κ_s is the effective heat conductance of the system and the exponent n is close to unity. In the current analysis of the

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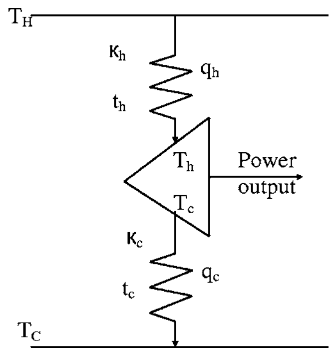

FIG. 1. Schematic of the endoreversible engine.

endoreversible heat engine we assume such a heat transfer law and we derive the efficiency of the engine at maximum power production. Consider an endoreversible heat engine which works between two heat reservoirs, the hot side at T_H and the cold side at T_C (see Fig. 1). The total amount of heat transferred from the hot reservoir to the engine's working fluid at T_h during the contact time t_h is given by

$$Q_H = \kappa_h t_h (T_H - T_h)^n. \tag{3}$$

Similarly for the cold side we have

$$Q_C = \kappa_c t_c (T_c - T_C)^n. \tag{4}$$

For reasons of energy conservation the work produced in a complete cycle is

$$W = Q_H - Q_C, \tag{5}$$

while the second law applied to the interior reversible engine yields

$$\Delta S = \frac{Q_H}{T_h} - \frac{Q_C}{T_c} = 0. \tag{6}$$

The independent variables may be chosen in many different ways, e.g., $x = T_H - T_h$ and $y = T_c - T_C$ (as was chosen by Ref. 6), or T_h and T_c with some choices more convenient than others. In the present study we will use the dimensionless time division of the hot side, $s = t_h/\tau$, and the thermodynamic efficiency, $\eta = W/Q = 1 - T_c/T_h$, where τ is the total cycle time, since this choice of independent variables simplifies the analysis. Introducing the last set of variables into the first and second laws of thermodynamics, the power production takes the form

$$w = \frac{W}{\tau} = \frac{\kappa_h T_H^n \eta(\eta_c - \eta)^n}{(1-\eta)^n \{(1/s)^{1/n} + (1-\eta)^{1/n-1}[\kappa/(1-s)]^{1/n}\}^n}, \tag{7}$$

where $\eta_c = 1 - T_C/T_H$ is the Carnot efficiency of the complete endoreversible engine, and $\kappa = \kappa_h/\kappa_c$.

To find the maximum power we differentiate with respect to s and η. The differentiation with respect to s leads to

$$\frac{1}{s} = 1 + \kappa^{1/(n+1)}(1-\eta)^{(1-n)/(1+n)}, \tag{8}$$

so that the power output expression simplifies to

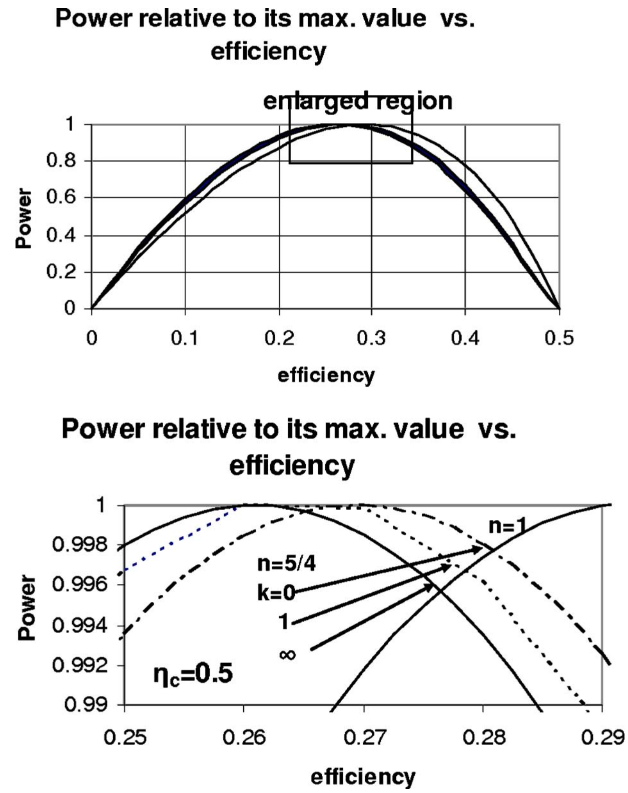

FIG. 2. Power output, relative to its maximum value, as a function of engine efficiency η for $n=5/4$ and a heat reservoir temperature ratio of 0.5 for different values of heat conductance ratio $\kappa = \kappa_h/\kappa_c$. Frame (a) covers the complete range of η, while frame (b) is a magnification of the maximum power region. For comparison, the case $n=1$ is plotted.

$$w = \frac{\kappa_h T_H^n \eta(\eta_c - \eta)^n}{(1-\eta)^n[1 + \kappa^{1/(1+n)}(1-\eta)^{(1-n)/(1+n)}]^{1+n}}. \tag{9}$$

The power vanishes at the two extremes: the open circuit limit where $\eta = \eta_c$ and the short circuit limit $\eta = 0$. The maximum power point is situated between these two extremes. In Fig. 2 we plot the power relative to its maximum value versus efficiency η for $\eta_c = 0.5$, $\kappa = 1$, and two values of n: 1 and 5/4. The maximum power point for $n = 5/4$ is shifted to the left when compared to $n = 1$. This maximum power point slightly depends on κ, as demonstrated by Fig. 2(b).

In order to find the efficiency at maximum power η_w we need to differentiate with respect to η. The result of this process leads to a nonlinear equation of η,

$$\eta^2 - \eta[1 + n + \eta_c(1-n)] + \eta_c + \frac{\eta(\eta_c - \eta)(1-n)\kappa^{1/(1+n)}(1-\eta)^{(1-n)/(1+n)}}{1 + \kappa^{1/(1+n)}(1-\eta)^{(1-n)/(1+n)}} = 0. \tag{10}$$

In principle this equation can be solved numerically for given κ, n, and η_c; however, it is more instructive to find approximate solutions for special situations. In the following section we will consider three different cases: (1) n values close to unity, i.e., $n = 1 + \varepsilon$ where ε is a small number, for which we will find an approximate expression for the efficiency at maximum power; (2) $\kappa = 0$; and (3) $\kappa \to \infty$. The last two cases will be solved exactly for the efficiency and the

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resulting expressions will serve as a check for the approximation derived for the case 1.

III. CONVECTION

We restrict n to be close to unity, $n=1+\varepsilon$ where ε is a small number which is representative of convective heat transfer. For this special case we find an approximate analytic expression for the efficiency at maximum power, based on the expansion in powers of ε,

$$\eta_w = \eta_0 + \eta_1\varepsilon + \cdots . \tag{11}$$

By substitution of this power expansion into the derived equation for η [Eq. (10)], and collecting terms of zeroth and first orders, we find

$$\eta_0 = 1 - \sqrt{1-\eta_c} \tag{12}$$

and

$$\eta_1 = -\frac{\eta_0}{2(1-\eta_0)}\left[1-\eta_c+\frac{\sqrt{\kappa}(\eta_c-\eta_0)}{\sqrt{\kappa}+1}\right] = (1-\phi)\left[\phi+\frac{\sqrt{\kappa}(1-\phi)}{\sqrt{\kappa}+1}\right], \tag{13}$$

where $\phi=\sqrt{1-\eta_c}$. The principal part of the efficiency at maximum power, η_0, is the Curzon-Ahlborn expression in the case of Newtonian heat transfer law $n=1$.[2] The first order correction η_1 is a small negative addition to the Curzon-Ahlborn expression. The effect of the distribution of heat conductance κ is found only in the correction term η_1. Thus the efficiency at maximum power is quite insensitive to the value κ.

Let us consider two specific cases in more detail, $\kappa=0$ corresponding to perfect conduction to the cold reservoir and $\kappa\to\infty$ corresponding to perfect heat conduction to the hot reservoir. The derived exact results will help us to check the accuracy of the approximate expression Eqs. (11)–(13). When $\kappa=0$ the exact Eq. (10) for η reduces to a quadratic, and its solution is

$$\eta_{(\kappa=0)} = \frac{1}{2}\{1+n+(1-n)\eta_c - \sqrt{[1+n+(1-n)\eta_c]^2-4\eta_c}\}. \tag{14}$$

Similarly, when $\kappa\to\infty$, Eq. (10) reduces again to a quadratic with the solution

$$\eta_{(\kappa\to\infty)} = \frac{1+n-\sqrt{(1+n)^2-4n\eta_c}}{2n}. \tag{15}$$

As a numeric example let us take $\eta_C=0.5$ corresponding to, e.g., $T_H=600$ K and $T_C=300$ K, $n=5/4$ corresponding to natural heat convection heat transfer,[8] and the two extremes of κ. For $\kappa=0$ the exact efficiency at maximum power from Eq. (14) is $\eta_w=0.269$ and the approximate from Eqs. (11)–(13) is 0.268. For $\kappa\to\infty$ the corresponding exact calculation leads to the value of 0.259 while the approximation gives 0.256. It is clear that the approximate Eqs. (11)–(13) provide a good representation. These maxima may be seen in Fig. 2(b).

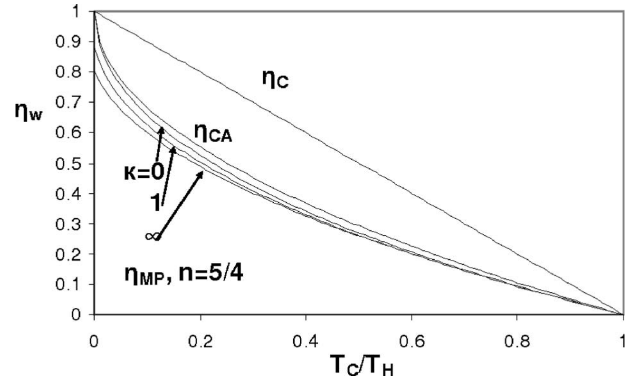

FIG. 3. Efficiency as a function of reservoir temperature ratio (T_C/T_H) for $n=5/4$ and different values of $\kappa=\kappa_h/\kappa_c$. For comparison the Carnot efficiency (η_C) and Curzon-Ahlborn (η_{CA}) efficiency are plotted on the same plot.

Figure 3 depicts the efficiency at maximum power production, η_w for the full range of possible reservoir temperature ratios. For comparison the Carnot and Curzon-Ahlborn efficiencies are included. It is seen that convective heat flow decreases the engine efficiency compared to Newtonian heat flow [Curzon-Ahlborn (CA)], most for thermal resistance on the cold branch.

IV. CONCLUSIONS

In the present analysis we considered an endoreversible heat engine with convective heat transfer such that the heat transfer takes place according to the temperature difference to the power n with n close to unity, $n=1+\varepsilon$ where ε is a small number.

Following an expansion procedure we found an approximate analytic expression for the efficiency at maximum power production which consists of a principal part equal to the Curzon-Ahlborn expression reduced by a small contribution which depends weakly on the heat conductance ratio κ. The approximate expression was checked against exact expressions for the extreme cases $\kappa=0$ and $\kappa\to\infty$ and found to be in agreement to within 1%. From the derived expression for the efficiency at maximum power it is clear that this efficiency is quite insensitive to the ratio κ.

The present study improves the analysis of this heat engine over the analysis given by Ref. 6 in two ways: (1) in the present study we have to solve one nonlinear equation for the efficiency, while in Ref. 6 two nonlinear coupled equations had to be solved simultaneously, and (2) no data fitting is needed as was done in Ref. 6.

In the present analysis, we retain complete generality of the model, and by proper choice of the independent variables, we derive one nonlinear equation for the efficiency at maximum power production. In principle this equation could be solved numerically, but it is more illustrative to find approximate analytical expression for special cases. While for the extreme cases of $\kappa=0$ and $\kappa\to\infty$ we derived exact

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analytical expressions for the efficiency. These expressions served as check for approximate formula Eqs. (11)–(13).

ACKNOWLEDGMENTS

One of the authors (M.H.) would like to thank the Nachemsohn Foundation for its financial support and the Ørsted Laboratory, University of Copenhagen for partial support and its hospitality.

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